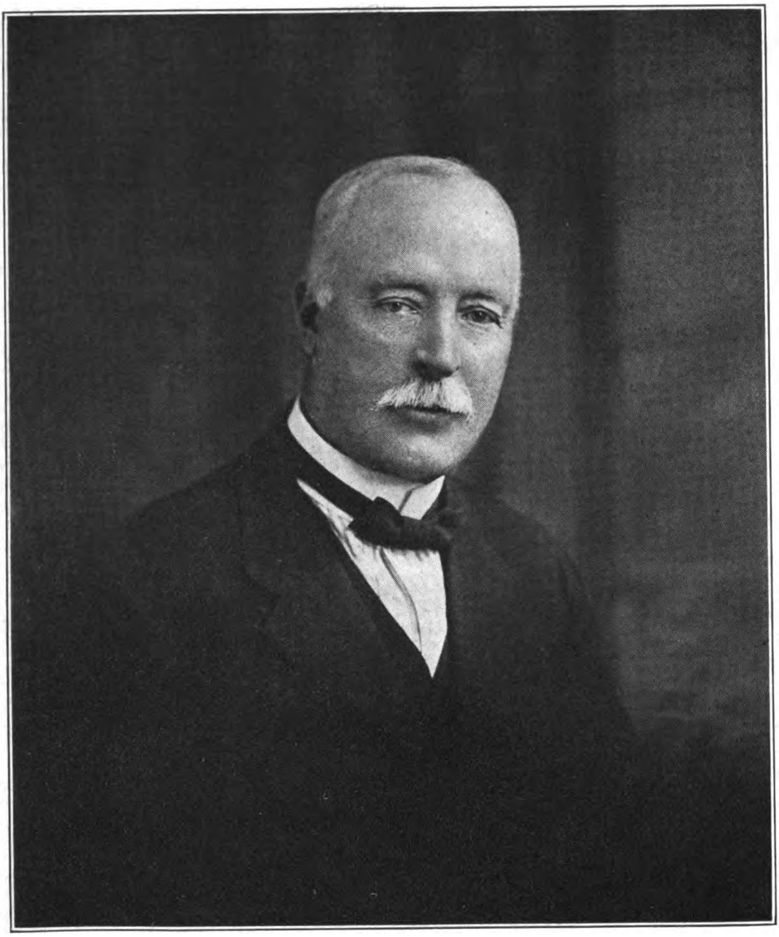

FRANK NELSON COLE

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE

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By order of the Council of the Society, this volume is dedicated to

FRANK NELSON COLE

in appreciation of his devotion to the Society during his twenty-six years as Secretary and in recognition of his efficient leadership in the editorial work of the Bulletin for the past twenty-four years.

BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

THE SEATTLE MEETING OF THE SAN FRANCISCO SECTION.

A special meeting of the San Francisco Section of the American Mathematical Society was held at Seattle, on Thursday and Friday mornings, June 17–18, in connection with the meetings of the Pacific division of the American Association for the Advancement of Science. Professor Blichfeldt, chairman of the Section, presided at the Thursday morning session, Professor Moritz temporarily filling the chair during the presentation of Professor Blichfeldt's paper. At the Friday morning session Professor Moritz presided. In the absence of Professor Bernstein, Professor Bell acted as temporary secretary.

The attendance included the following twelve members of the Society:

Professor E. T. Bell, Professor H. F. Blichfeldt, Professor A. F. Carpenter, Professor E. C. Colpitts, Professor G. I. Gavett, Dr. G. F. McEwen, Professor W. E. Milne, Professor R. E. Moritz, Professor L. I. Neikirk, Dr. L. L. Smail, Dr. A. R. Williams, Professor R. M. Winger.

The following papers were presented at this meeting:

(1) Professor W. F. Durand: "Some points of contact between mathematics and applied science."

(2) Professor H. F. Blichfeldt: "On the approximate representation of irrational numbers, and the theory of geometry of numbers."

(3) Professor E. T. Bell: "An arithmetical dual of Kummer's quartic surface."

(4) Dr. T. H. Gronwall: "On the Minkowski volume and surface theory."

(5) Professor A. F. Carpenter: "Congruences associated with a ruled surface."

(6) Professor FLORIAN CAJORI: "Moritz Cantor, the historian of mathematics."

(7) Professor L. I. NEIKIRK: "The functional variable. Second and higher derivatives."

(8) Dr. L. L. SMAIL: "Notes on some points in the theory of summable series."

(9) Dr. L. L. SMAIL: "Bibliography of the theory of summable series."

(10) Professor E. T. BELL: "An image in four dimensional lattice space of the theory of the elliptic theta functions."

(11) Professor E. T. BELL: "Certain arithmetical consequences of the equation of three terms in elliptic functions."

(12) Professor E. T. BELL: "Systems of higher determinate equations."

(13) Professor W. E. MILNE: "Infinite systems of vectors."

Professor Durand was introduced by Professor Blichfeldt. In the absence of the authors, Dr. Gronwall's paper was read by title and Professor Cajori's was read by Professor Blichfeldt. Professor Bell's last three papers were also read by title. Abstracts of the papers follow below.

1. Professor Durand's paper first discusses the types of problem that arise in applied science with especial reference to their mathematical requirements, and suggests certain lines for the classification of such problems. Following this general view of the subject, mention is made of some specific problems for the treatment of which adequate mathematical means are still lacking. This section of the paper is intended to suggest to mathematicians needs for the further development of mathematical methods and means of analysis in order to treat adequately actual problems that arise in various fields of applied science.

2. Professor Blichfeldt discusses certain fundamental theorems in the geometry of numbers, after which the approximate solution in integers of a system of linear equations is taken up, in particular of the system $x_i + a_i - x_1x_{n+1} + b_i = 0$ $(i = 1, 2, \cdots, n)$. The results of Kronecker, Hermite and Minkowski are compared, and a recent theorem on this system by the author is stated.

3. If E is a homogeneous algebraic equation in x, y, z, w, it represents a surface S when the variables are point coordi-

nates; if the variables are interpreted arithmetically, E can be made to represent a system S' of theorems concerning integers. When S implies S' and S' implies S, Professor Bell calls S, S' arithmetical duals of each other, and determines an S' for Kummer's S, such that all the geometrical properties of S imply and are implied by systems of arithmetical theorems, which in turn may be interpreted in terms of lattice configurations lying upon two concentric spheres in space of sixteen dimensions.

5. Paul Serret* has proved the theorem "the double-ratio of the four points in which any generator of a ruled surface intersects four fixed asymptotic curves is constant." Analogously, any four directrix curves of a ruled surface which cut all the generators in four points of constant double-ratio may be called a Serret set. Let C_y, C_z, C_μ, C_ν be any four directrix curves of a ruled surface S, P_y, P_z, P_μ, P_ν the points where they cut any generator g, and T_y, T_z, T_μ, T_ν the tangents to C_y, C_z, C_μ, C_ν at these points. T_y, T_z, T_μ determine a quadric $K\theta$. Professor Carpenter's paper proves that T_ν will lie upon $K\theta$ and be a ruling of the same kind as T_y, T_z, T_μ if and only if C_y, C_z, C_μ, C_ν constitute a Serret set, and that there is a one-parameter family of such quadrics $K\theta$ associated with each generator g of S. The generator g belongs to one regulus of each quadric $K\theta$. The paper next considers one of the quadrics $K\theta$ whose equation is invariant in form. One of these quadrics is associated with each generator of S, and this family of quadrics gives rise to two congruences Δ_1 and Δ_2 whose properties are investigated.

6. Professor Cajori gives a biographical sketch of Moritz Cantor and an estimate of his work as a historian.

7. Volterra and others have given implicit definitions for a function of a line and its first derivative. Professor Neikirk, following the method of one of his previous papers, gives explicit definitions of second and higher derivatives. Among other results, he gives the following classification of functions of a functional variable: (1) The first derivative is a point function. The second derivatives are zero or infinite. For functions of this class the methods of this paper may be

* P. Serret, Théorie nouvelle géométrique et mécanique des Lignes à double Courbure, Paris, Bachelier, 1860.

used to derive the second variation as used in the calculus of variations. (2) The first derivative is a function of a functional variable. The second derivatives are the results of repeated operations. (3) The first derivative is a mixed, point and functional variable, function. The second derivatives are the results of the repeated operations except for the case F_{ii}, which is infinite.

8. In Dr. Smail's first paper, a number of minor gaps in the systematic treatment of the theory of summable divergent series are filled in, and new proofs of some familiar theorems are given.

9. Dr. Smail's second paper gives a bibliography of the theory of summable series.

10. In this paper Professor Bell proves eleven geometrical theorems concerning point configurations upon a sphere in lattice space of four dimensions which imply, and are implied by, the theory of the elliptic theta functions.

11. Extending Liouville's theorems in the fifth of his memoirs on general formulas useful in the theory of numbers, Professor Bell shows that similar general formulas exist for a pair of bilinear forms each in four variables, and finds a complete set of 57 such formulas.

12. Professor Bell shows in this paper by two detailed examples how the general formulas of the kind given in the preceding paper may be used to derive information concerning the number of solutions of certain systems of indeterminate equations of any degree.

13. In this paper Professor Milne shows that corresponding to every infinite set of vectors of finite norm in infinitely many dimensions there exists an infinite set of constants λ_i $(i = 1, 2, \cdots)$ with the following properties: The necessary and sufficient condition that (*a*) the system be orthogonal is that $\lambda_i = 1$ for every i; (*b*) the system have an adjoint is that $\lambda_i > 0$ for every i; (*c*) the system be essentially linearly dependent is that $\lambda_i = 0$ for some i. Applications are made to infinite matrices and to infinite sets of linear equations.

E. T. Bell,
Acting Secretary of the Section.

NOTE ON A GENERALIZATION OF A THEOREM OF BAIRE.

BY PROFESSOR E. W. CHITTENDEN.

(Read before the American Mathematical Society September 8, 1920.)

It is the purpose of this note to call attention to the following generalization of a celebrated theorem of Baire:

THEOREM: *Let P be a perfect set in a complete, separable, metric space S.* Then a necessary and sufficient condition that a function defined on P be of class 1 in the Baire classification of functions is that the function be at most point-wise discontinuous on every perfect subset of P.*

The necessity of the condition has been established by Hausdorff.†

A satisfactory proof of this theorem can be obtained from a proof given by Vallée-Poussin‡ for the corresponding theorem in space of n dimensions by making appropriate changes in terminology, since Vallée-Poussin makes no essential use of the special properties of space of n dimensions in his argument. The methods to be followed in generalizing this proof of Vallée-Poussin are sufficiently indicated in the treatise of Hausdorff already cited and in the memoirs of Fréchet.§

There is, however, one point which requires special consideration. Vallée-Poussin introduces an auxiliary function with the following definition: If M is a point of S; H_1, H_2, $\cdots$, H_n are closed sets such that no two have a common element; δ_1, δ_2, $\cdots$, δ_n are the respective distances of H_1, H_2, $\cdots$, H_n from M; and a_1, a_2, $\cdots$, a_n are constants; then

* See Hausdorff, Grundzüge der Mengenlehre, Veit und Co., Leipzig, 1914, pp. 211, 315. The word "vollständig" is here translated "complete." A complete set admits a generalization of the theorem of Cauchy in the sense of Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matem. di Palermo*, vol. 22 (1906), pp. 1–74.

† Hausdorff, loc. cit., p. 388.

‡ Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire, Gauthier-Villars, Paris, 1916, pp. 105–125.

§ As an example of the application of the theory of transfinite ordinal numbers in the present situation, see Fréchet, "Les ensembles abstraits et le calcul fonctionnel," *Rendiconti del Circolo Matem. di Palermo*, vol. 30 (1910), pp. 5–10.

$$\varphi(M) = \frac{a_1/\delta_1 + a_2/\delta_2 + \cdots + a_n/\delta_n}{1/\delta_1 + 1/\delta_2 + \cdots + 1/\delta_n},$$

except when M is in H_i; then $\varphi(M) = a_i$ ($i = 1, 2, 3, \cdots, n$).*

The function $\varphi(M)$ is continuous in space of n dimensions. I wish to show that it is continuous in the space S. It is sufficient to show that the distance δ of a point M from a closed set H is a continuous function of M, vanishing on H.

Let (A, B) denote the distance between two points A, B of the metric space S. Then we have the following fundamental property of distance: If A, B, C are any three points of S,

$$(A, B) \leqq (A, C) + (C, B).\dagger$$

If H is a closed set the distance (M, X), where X is in H, has a minimum δ at a point X_0. We call $\delta = (M, X_0)$ the distance from M to H. Let M' be any other point of H, and $\delta' = (M', X_0')$ be the corresponding distance from H. Then we have

$$\delta = (M, X_0) \leqq (M, X_0') \leqq (M, M') + (M', X_0')$$
$$\leqq (M, M') + \delta'.$$

Similarly, $\delta' \leqq \delta + (M, M')$. Therefore $|\delta - \delta'| \leqq 2(M, M')$, and the continuity of δ is established.

UNIVERSITY OF IOWA,
IOWA CITY, IOWA.
February 28, 1920.

CERTAIN ITERATIVE CHARACTERISTICS OF BILINEAR OPERATIONS.

BY DR. NORBERT WIENER.

Introduction.

IN a recent paper‡ the author has developed the necessary and sufficient condition that a bilinear operation in two variables should generate by iteration all rational operations with

* Vallée-Poussin, loc. cit., p. 118.
† Hausdorff, loc. cit., p. 211. See also Fréchet, "Les notions de limite et de distance," *Transactions Amer. Math. Society*, vol. 19 (1918), p. 54.
‡ *Annals of Mathematics*, vol. 21, No. 3 (March, 1920), pp. 157–165.

rational coefficients. This is a particular case of the general problem of determining just what operations any given bilinear operation will generate by iteration. While the complete solution of this problem is still to be accomplished, it is the purpose of this paper to develop methods of attack which will yield, in particular, an important necessary condition that each of two operations generate the other by iteration.

Definitions.

A bilinear operation is an operation $\varphi(x, y)$ of the form

$$\frac{A_\phi + B_\phi x + C_\phi y + D_\phi xy}{E_\phi + F_\phi x + G_\phi y + H_\phi xy}.$$

In this paper the coefficients will be supposed to be rational, ∞ will be considered a proper value and argument for a bilinear operation, and if $\varphi(x_1, y_1)$ is indeterminate as it stands, it shall be given the value $\lim_{x \doteq x_1,\, y \doteq y_1} \varphi(x, y)$ if this is determinate.

A bilinear operation will be said to be reduced if A_ϕ, $B_\phi + C_\phi$, D_ϕ, E_ϕ, $F_\phi + G_\phi$, and H_ϕ form a relatively prime set of integers.

Δ_ϕ is defined as

$$\begin{vmatrix} \begin{vmatrix} A_\phi & B_\phi + C_\phi \\ E_\phi & F_\phi + G_\phi \end{vmatrix} & \begin{vmatrix} A_\phi & D_\phi \\ E_\phi & H_\phi \end{vmatrix} \\ \begin{vmatrix} A_\phi & D_\phi \\ E_\phi & H_\phi \end{vmatrix} & \begin{vmatrix} B_\phi + C_\phi & D_\phi \\ F_\phi + G_\phi & H_\phi \end{vmatrix} \end{vmatrix}.$$

The following definitions are made:

$$J_\phi(u, v) = A_\phi v^2 + (B_\phi + C_\phi)uv + D_\phi u^2,$$

$$K_\phi(u, v) = E_\phi v^2 + (F_\phi + G_\phi)uv + H_\phi u^2,$$

$$L_\phi = vJ_\phi - uK_\phi.$$

We shall term the roots of $L_\phi(u, 1) = 0$, $r_1(\varphi)$, $r_2(\varphi)$, $r_3(\varphi)$. We shall call a root of $L_\phi(u, 1) = 0$ free if it is not a simultaneous solution of $J_\phi(u, 1) = 0$ and of $K_\phi(u, 1) = 0$. If $\Delta_\phi \neq 0$, and $r_1(\varphi) \neq r_2(\varphi) \neq r_3(\varphi)$, $r_1(\varphi) \neq r_3(\varphi)$, φ will be said to be general, otherwise special. Clearly if φ is general r_1, r_2, and r_3 are all free.

An iterative field is a set of functions of n variables which is closed with respect to the following operations:

(1) permutation of any two arguments;

(2) substitution of two identical arguments for two different arguments;

(3) substitution of a function of the set for any argument.

Two operations are said to be iteratively equivalent if any iterative field containing either contains the other.

An iterative characteristic of an operation is a property which belongs to all operations iteratively equivalent to that operation.

Theorems.

I. *The free roots of* $L_\phi(u, 1) = 0$ *constitute iterative characteristics of* φ.

For $L_\phi(u, 1) = 0$ is equivalent to $\varphi(u, u) = u$, unless u is a simultaneous root of $J_\phi(u, 1) = 0$ and of $K_\phi(u, 1) = 0$. Now, $\varphi(u, u) = u$ is a property of φ invariant under iteration. If r is a root of $L_\phi(u, 1) = 0$ that is not free, then in general $\varphi(r, r) = r$ is not significant and determinate. If this latter equation is significant, let us subject our number system to a linear transformation that changes r to ∞. It may readily be seen that two operations will retain their iterative relationships unchanged under such transformations, that roots of $L_\phi(u, 1) = 0$ will remain roots, and that free roots will remain free. φ will then become a ψ of the form $A + Bx + Cy$. This operation can only be iteratively equivalent to operations of its own sort, for all of which ∞ is a root that is not free. This completes the proof of our theorem. As a corollary we obtain

II. *No general bilinear operation is equivalent to any special bilinear operation.*

III. *If* φ *and* ψ *are reduced equivalent general bilinear operations, then*

$$A_\phi = A_\psi,$$

$$B_\phi + C_\phi - E_\phi = B_\psi + C_\psi - E_\psi,$$

$$F_\phi + G_\phi - D_\phi = F_\psi + G_\psi - D_\psi,$$

$$H_\phi = H_\psi.$$

Lemma 1. *If* φ *is reduced,* u *and* v *can be chosen relatively prime in such a manner that* $J_\phi(u, v)$ *is prime to* $K_\phi(u, v)$.

Proof. Let N be the H.C.F. of A_ϕ, $B_\phi + C_\phi$, and D_ϕ. Let M be the product of all prime factors in Δ_ϕ but not in N. Let P be the product of all the primes in M, but not in A_ϕ nor in D_ϕ. Let Q be the product of all primes in M and A_ϕ, but not in D_ϕ. Let R be the product of all primes in M and D_ϕ, but not in A_ϕ. Let S be the product of all primes in N, but neither in E_ϕ nor in H_ϕ. Let T be the product of all primes in N and E_ϕ, but not in H_ϕ. Let U be the product of all primes in N and H_ϕ, but not in E_ϕ. Let $u = PRSU$, $v = QT$. We get as a result that J_ϕ is prime to M, while K_ϕ is prime to N. Now, it may readily be shown that any factor common to J_ϕ and K_ϕ belongs to Δ_ϕ. Since, however, any prime factor of Δ_ϕ belongs either to M or to N, it is either prime to J_ϕ or to K_ϕ.*

Lemma 2. *If a and b are any two numbers not both containing a prime p and if c and d, e and f, are couples of the same sort, while*

$$ad - bc \equiv 0 \pmod{p^k}, \qquad af - be \equiv 0 \pmod{p^k},$$

then

$$cf - de \equiv 0 \pmod{p^k}.$$

This lemma, whose proof is immediate, enables us to classify fractions into sets modulo p^k. We shall say that

$$a/b \equiv c/d \pmod{p^k}$$

if

$$ad - bc \equiv 0 \pmod{p^k}$$

while a and b do not contain p in common, nor do c and d.

Lemma 3. *If u/v and w/z are in their lowest terms, and*

$$u/v \equiv w/z \pmod{p^k},$$

then if φ is any bilinear operation such that

$$\varphi(u/v,\, u/v) \equiv u/v \pmod{p^k},$$

it follows that

$$\varphi(w/z,\, w/z) \equiv w/z \pmod{p^k}.$$

The proof of lemma 3 offers no difficulty.

Combining lemmas 2 and 3, it is at once clear that the equation

$$\varphi(u/v,\, u/v) \equiv u/v \pmod{p^k}$$

for a given u/v and p^k is an iterative characteristic of φ.

* Cf. Mathews, Theory of Numbers, vol. 1, p. 132.

We are now in a position to take up the proof of Theorem III. Clearly $\varphi(u/v, u/v)$ is equal to $J_\phi(u, v)/K_\phi(u, v)$. Let u and v be two numbers such that $J_\phi(u, v)$ and $K_\phi(u, v)$ are prime to one another. By lemma 1, such numbers exist. Then to say that

$$J_\phi(u, v)/K_\phi(u, v) \equiv u/v \pmod{p^k}$$

is equivalent to saying

$$L_\phi(u, v) \equiv 0 \pmod{p^k},$$

or that p^k is a factor of $L_\phi(u, v)$. If φ is iteratively equivalent to ψ, clearly we must have

$$L_\psi(u, v) \equiv 0 \pmod{p^k}.$$

That is, every prime power that divides L_ϕ must also divide L_ψ, or in other words, L_ϕ is a factor of L_ψ for a particular set of values of u and v. However, since by Theorem I the equations $L_\phi(u, 1) = 0$ and $L_\psi(u, 1) = 0$ have the same three distinct roots, L_ψ is a constant multiple of L_ϕ. Hence L_ϕ is always a factor of L_ψ. Likewise, L_ψ is always a factor of L_ϕ. Hence, apart from a possible difference of sign, L_ϕ and L_ψ are identical. This proves Theorem III.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
April 16, 1920.

NECESSARY AND SUFFICIENT CONDITIONS THAT A LINEAR TRANSFORMATION BE COMPLETELY CONTINUOUS.

BY PROFESSOR CHARLES ALBERT FISCHER.

(Read before the American Mathematical Society December 31, 1919.)

A LARGE part of the Fredholm theory of integral equations has been derived for the equation

$$h(x) = f(x) + \lambda T(f),$$

where $T(f)$ is a completely continuous linear transformation.* It has also been proved that every linear transformation, that

* F. Riesz, *Acta Mathematica*, vol. 41 (1918), pp. 71–98.

is, linear functional depending on a parameter, is equal to a Stieltjes integral of the form*

$$(1) \qquad T(f) = \int_a^b f(y) d_y K(x, y).$$

In a former paper† I have found a necessary and some sufficient conditions that a linear transformation be completely continuous, and thus proved that this part of the Fredholm theory applies to Stieltjes integral equations of certain types. In the present paper will be found conditions which are both necessary and sufficient. The Volterra theory of Stieltjes integral equations has been discussed by Evans in his Cambridge Colloquium Lectures, 1916.

§ 1. *Preliminary Theorems.*

A set of functions is said to be compact if every infinite sequence of them contains an infinite subsequence which is uniformly convergent, and a transformation is completely continuous if it changes every bounded set of functions into a compact set.

The class $[f(y)]$ will be composed of all functions defined on the interval (a, b) which can be approached by monotone sequences, beginning with the class of all continuous functions. The definition of the Stieltjes integral has been extended by Young‡ to apply to such functions.

The function $K(x, y)$ will be assumed to satisfy the equation

$$(2) \qquad K(x, y) = K(x, y + 0).$$

This will not affect the value of the integral (1), and when this equation is satisfied, $V_y K(x, y)$, that is, the variation of K considered as a function of y, is the least upper bound of $T(f)/\max|f|$.§ It follows that if $V_y K(x, y)$ is not bounded uniformly in x, the transformation (1) cannot be completely continuous, because in that case there would be a bounded set of f's for which the set $[T(f)]$ would not be bounded and therefore could not be compact.

* F. Riesz, *Annales Scientifiques de l'Ecole Normale*, ser. 3, vol. 31 (1914), pp. 9–14.

† This BULLETIN, vol. 25 (1919), p. 447.

‡ Young, *Proc. London Math. Society*, vol. 13 (1914), p. 109, or Daniell, *Annals of Mathematics*, vol. 19 (1918), p. 279.

§ Riesz, loc. cit., *Annales*; or Fischer, *Annals of Mathematics*, vol. 19 (1917), pp. 38–40.

It will now be proved that when V_yK is bounded uniformly, every sequence of x's must contain a subsequence for which $K(x_n, y)$ approaches a uniquely determined function of y when n becomes infinite.

The positive constants $\epsilon_1, \epsilon_2, \cdots$ will be chosen so as to approach zero as a limit, and it will be assumed that $V_yK < M$. If there were a sequence of x's no subset of which would make the limit of $K(x_n, y)$ unique, there would be no subset which would make the difference between the largest and smallest limit points of $K(x_n, y)$ less than or equal to ϵ_1 uniformly in y, or else there would be such a subset which will be designated as $x_1^{(1)}, x_2^{(1)}, \cdots$. Then there would be no subset of the $x^{(1)}$'s which would make the same difference less than ϵ_2 uniformly, or else there would be such a subsequence, $x_1^{(2)}, x_2^{(2)}, \cdots$. It would not be possible to proceed in this way and get an infinite number of sequences, $x_1^{(i)}, x_2^{(i)}, \cdots$, each a subset of the proceeding, and such that the difference between the largest and smallest limit points of $K(x_n^{(i)}, y)$ would always be less than ϵ_i, because if this could be done the limit of $K(x_n^{(n)}, y)$ would be unique. Consequently there would have to be a subset of the original x's, $\bar{x}_1, \bar{x}_2, \cdots$, and a $k > 0$, such that no subsequence of the $\bar{x}$'s could make the difference between the largest and smallest limit points of $K(\bar{x}_n, y)$ less than or equal to k uniformly in y.

When a bounded set of functions is given there is always a subset of them that converges at any previously given denumerable set of points.* It follows that there would be a non-denumerable set of values of y for which the difference between limit points of $K(\bar{x}_n, y)$ would be greater than k, for a given sequence of the $\bar{x}$'s, since if there were only a denumerable set of such y's, the subset of the $\bar{x}$'s which made the above limit unique for these y's would make the difference between limit points of $K(\bar{x}_n, y)$ less than or equal to k for all values of y.

A subset of the $\bar{x}$'s could be taken such that $K(\bar{x}_n, y)$ would converge at a dense set of y's, and the other $\bar{x}$'s dropped. Then if y_1 were a point where two of its limit points differed by more than k, there would be two $\bar{x}$'s which would make K differ by more than k at y_1, while they made it differ by an arbitrarily small amount at a point arbitrarily near to y_1, and consequently V_yK would be greater than $k/2$ in the neigh-

* Riesz, loc. cit., *Acta*, p. 93.

borhood of y_1 for at least one of these values of x. To state this more definitely, if a sequence $\delta_1, \delta_2, \cdots$ were given, approaching zero, there would be a sequence of $\bar{x}$'s which would satisfy the inequalities

$$V_yK(\bar{x}_n, y)\Big|_{y_1-\delta_n}^{y_1+\delta_n} > \frac{k}{2} \qquad (n = 1, 2, \cdots).$$

The other $\bar{x}$'s could then be dropped. Similarly there would be a y_2 and a subset of the remaining $\bar{x}$'s which would have the same property with respect to y_2, and the other $\bar{x}$'s could again be dropped. Proceeding in this way, the points $y_1, y_2, \cdots, y_N$ $(N > 2M/k)$ and a sequence of $\bar{x}$'s would be determined which would satisfy the inequalities

$$V_yK(\bar{x}_n, y)\Big|_{y_i-\delta_n}^{y_i+\delta_n} > \frac{k}{2} \qquad (i = 1, 2, \cdots, N).$$

If n were then taken so large that the intervals $(y_i - \delta_n, y_i + \delta_n)$ did not overlap, the inequalities

$$V_yK(\bar{x}_n, y) \geqq \sum_{i=1}^{N} V_yK(\bar{x}_n, y)\Big|_{y_i-\delta_n}^{y_i+\delta_n} > \frac{Nk}{2} > M,$$

would be satisfied. But this is contrary to the hypothesis that $V_yK < M$.

This completes the proof of the theorem: *If V_yK is bounded uniformly in x, when a denumerable set of the x's is given, there must be a subsequence of them which makes $K(x_n, y)$ approach a unique limit as n becomes infinite.*

It will now be proved that if a set of functions converges, the variation of the limiting function cannot be greater than the limit of the variations of any sequence of the given functions.

If this were not the case there would be a sequence $\varphi_1(y)$, $\varphi_2(y), \cdots$, approaching a function $\varphi(y)$, and a $k > 0$, which would satisfy the inequalities

$$V\varphi(y) > V\varphi_n(y) + k \qquad (n = 1, 2, \cdots).$$

Then there would have to be a finite set of points $y_0 = a < y_1 < y_2 < \cdots < y_N = b$, which would satisfy the inequalities

$$(3) \quad \sum_{i=0}^{N-1} |\varphi(y_{i+1}) - \varphi(y_i)| > V\varphi_n(y) + \frac{k}{2} \qquad (n = 1, 2, \cdots).$$

But if n is large enough the inequalities

$$|\varphi_n(y_i) - \varphi(y_i)| < \frac{k}{4N} \qquad (i = 0, 1, 2, \cdots, N),$$

would be satisfied, and consequently inequality (3) would imply that

$$\sum_{i=0}^{N-1} |\varphi_n(y_{i+1}) - \varphi_n(y_i)| > V\varphi_n(y),$$

which is absurd.

§ 2. *Necessary and Sufficient Conditions.*

The following condition, which will be called condition A, will be proved to be necessary. Then a second condition, which will be called condition B, will be proved to be sufficient, and finally the two conditions will be proved to be equivalent.

CONDITION A. *A necessary and sufficient condition that the transformation* (1) *be completely continuous is that $V_y K(x, y)$ shall be bounded uniformly, and that when a sequence of x's is chosen in such a way that $K(x_n, y)$ converges, the equation*

$$\text{(4)} \qquad \underset{n=\infty}{\text{limit}}\, V_y[K(x_n, y) - \underset{n=\infty}{\text{limit}}\, K(x_n, y)] = 0$$

shall be satisfied.

If this condition is not satisfied there must be a sequence of x's for which $K(x_n, y)$ converges, and a $k > 0$, which satisfy the inequalities

$$\text{(5)} \qquad V_y[K(x_n, y) - \underset{n=\infty}{\text{limit}}\, K(x_n, y)] > k.$$

If the functions $K(x_n, y) - \underset{n=\infty}{\text{limit}}\, K(x_n, y)$ should also satisfy equations such as (2), there would be continuous functions $f_1(y), f_2(y), \cdots$, not greater than 1 in absolute value, which would satisfy the inequalities

$$\int_a^b f_m(y) d_y[K(x_m, y) - \underset{n=\infty}{\text{limit}}\, K(x_n, y)] > k \qquad (m = 1, 2, \cdots).$$

Since this is not necessarily the case, it will be proved instead that when x_m is given there must be an $m' \geqq m$, and a con-

tinuous $f_m(y)$, such that $|f_m| \leqq 1$, and

$$(6) \qquad \int_a^b f_m(y) d_y[K(x_{m'}, y) - \text{limit } K(x_n, y)] > \frac{k}{2}.$$

It follows from the last theorem in § 1, that when inequalities (5) are satisfied, when m is given there must be an $n > m$ which satisfies the inequality

$$V_y[K(x_m, y) - K(x_n, y)] > k.$$

Consequently there must be an $f_m(y)$ which satisfies the inequality

$$\int_a^b f_m(y) d_y[K(x_m, y) - K(x_n, y)] > k,$$

and at least one of the points x_m and x_n can be used for $x_{m'}$, in inequality (6).

The equation

$$\underset{n=\infty}{\text{limit}} \int_a^b f(y) d_y K(x_n, y) = \int_a^b f(y) d_y \text{ limit } K(x_n, y)$$

must be satisfied if $f(y)$ is continuous.* Then when $f_m(y)$ is given there must be a finite N such that

$$\left| \int_a^b f_m(y) d_y[K(x_n, y) - \text{limit } K(x_n, y)] \right| < \frac{k}{6} \qquad (n \geqq N).$$

It will then be possible to select a sequence $\xi_1, \xi_2, \cdots$ from the given x's, and a sequence of continuous f's, which satisfy the inequalities, $|f_i| \leqq 1$, and

$$\left| \int_a^b f_i(y) d_y[K(\xi_j, y) - \text{limit } K(x_n, y)] \right| < \frac{k}{6}$$

$$(i = 1, 2, \cdots; \; j = i + 1, i + 2, \cdots),$$

and

$$\int_a^b f_j(y) d_y[K(\xi_j, y) - \text{limit } K(x_n, y)] > \frac{k}{2}$$

$$(j = 1, 2, \cdots).$$

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* This follows from the proof of Bray's Theorem 3, *Annals of Mathematics*, ser. 2, vol. 20, pp. 180–181.

This implies that either

$$\left|\int_a^b [f_j(y) - f_i(y)]d_yK(\xi_j, y)\right| > \frac{1}{2}\left(\frac{k}{2} - \frac{k}{6}\right) = \frac{k}{6},$$

or else

$$\left|\int_a^b [f_j(y) - f_i(y)]d_y \text{ limit } K(x_n, y)\right| > \frac{k}{6}.$$

In either case there must be some value of x for which

$$|T(f_j) - T(f_i)| > \frac{k}{6}$$

$$(i = 1, 2, \cdots; \ j = i + 1, i + 2, \cdots),$$

and the transformation cannot be completely continuous. Therefore condition A is necessary.

CONDITION B. *A necessary and sufficient condition that the transformation* (1) *be completely continuous, is that when an* $\epsilon > 0$ *is chosen there must be a finite number of points* $\xi_1, \xi_2, \cdots, \xi_n$, *such that to every* x *in* (a, b) *there corresponds a* ξ_i *which satisfies the inequality*

$$V_y[K(x, y) - K(\xi_i, y)] < \epsilon. \tag{7}$$

If this condition is satisfied, and a decreasing sequence $\epsilon_1, \epsilon_2, \cdots$ is taken approaching zero, the ξ's corresponding to all the ϵ's can be arranged in one denumerable sequence. Every bounded set of f's must have a subset such that the functions

$$g_n(x) = T(f_n) = \int_a^b f_n(y)d_yK(x, y) \qquad (n = 1, 2, \cdots),$$

will converge at the denumerable set of points $\xi_1, \xi_2, \cdots$. This subset will now be proved uniformly convergent on the whole interval (a, b).

If an $\epsilon > 0$ is chosen there must be a finite N such that to every x on (a, b) there will correspond a ξ_i, $(i \leqq N)$, which satisfies inequality (7). Then if N' is so large that the inequalities

$$|g_n(\xi_i) - \underset{n=\infty}{\text{limit}}\, g_n(\xi_i)| < \epsilon \qquad (i \leqq N; \ n \geqq N'),$$

are satisfied, the inequalities

$$|g_n(x) - g_n(\xi_i)| \leqq \max|f| \cdot V_y[K(x, y) - K(\xi_i, y)] \qquad (n = 1, 2, \cdots),$$

and inequality (7) will imply that

$$|g_n(x) - g_{n+m}(x)| \leqq 2[\max|f| + 1]\epsilon \qquad (n \geqq N';\ m = 1, 2, \cdots).$$

Consequently these g's must be uniformly convergent, and the transformation must be completely continuous.

If condition B is not satisfied there must be an $\epsilon > 0$ and an infinite sequence of x's such that

$$V_y[K(x_m, y) - K(x_n, y)] \geqq \epsilon \qquad (m \neq n).$$

It follows from the principal theorem of the first section, that if V_yK is bounded uniformly, a subset of these x's must make $K(x_n, y)$ convergent. As no subset of them can satisfy equation (4), condition A cannot be satisfied. Therefore when A is satisfied B must be, and since A is necessary and B sufficient, B must be necessary and A sufficient also.

TRINITY COLLEGE,
HARTFORD, CONN.

ON THE RELATION OF THE ROOTS AND POLES OF A RATIONAL FUNCTION TO THE ROOTS OF ITS DERIVATIVE.

BY MR. BEN-ZION LINFIELD.

(Read before the American Mathematical Society December 30, 1919.)

1. F. Lucas, *Journal de l'Ecole polytechnique*, 1879, gave a mechanical proof of the following theorem:

The roots of the derivative of a cubic are the foci of the maximum ellipse inscribed in the triangle whose vertices are the roots of the cubic.

Professor Maxime Bôcher gave a simple proof of this theorem, *Annals of Mathematics*, volume 7, page 70, 1892. Here he made observations on the general theorem concerning the polynomial of nth degree and asked the question, "Could

not the first of these propositions be brought into connection with the focal properties of the higher plane curves?" Other demonstrations of Lucas's theorem have been given, but the writer is not aware of Professor Bôcher's question having been answered. It is the object of this note to answer this question by the following:

2. THEOREM. *The roots of the derivative of the rational function*

$$f(z) \equiv \prod_{i=1}^{n} (z - z_i)^{\mu_i}$$

are the multiple roots of $f(z)$ to one lower order and the foci of a curve of class $n - 1$ which touches each segment z_iz_j ($i \neq j$, $= 1, 2, \cdots, n$) at a point dividing it in the ratio of μ_i to μ_j.

3. Let

$$\alpha_i \equiv ux_i + vy_i + w \qquad (i = 1, 2, \cdots, n)$$

be the line equation of the point $P_i \equiv (x_i, y_i)$, where $z_i \equiv x_i + y_i\sqrt{-1}$ are the roots and poles of the rational function $f(z)$. Then,

$$\Pi \equiv \alpha_1\alpha_2\cdots\alpha_n = 0 \tag{1}$$

is the line equation of the n points x_i, y_i.

$$\varphi(u, v, w) \equiv \sum_{i=1}^{n} \mu_i \frac{\partial \Pi}{\partial \alpha_i} = 0 \tag{2}$$

is the tangential (line) equation of a curve of class $n - 1$ tangent to the segments P_iP_j.

The point of contact of an arbitrary tangent to (2) whose coordinates are u, v, w, is given by equations

$$\frac{X}{\frac{\partial \varphi}{\partial u}} = \frac{Y}{\frac{\partial \varphi}{\partial v}} = \frac{1}{\frac{\partial \varphi}{\partial w}}.$$

Also from (2)

$$\frac{\partial \varphi}{\partial u} = \Sigma(\mu_j x_i + \mu_i x_j)\frac{\partial^2 \Pi}{\partial \alpha_i \partial \alpha_j},$$

$$\frac{\partial \varphi}{\partial v} = \Sigma(\mu_j y_i + \mu_i y_j)\frac{\partial^2 \Pi}{\partial \alpha_i \partial \alpha_j},$$

$$\frac{\partial \varphi}{\partial w} = \Sigma(\mu_i + \mu_j)\frac{\partial^2 \Pi}{\partial \alpha_i \partial \alpha_j}.$$

Therefore the coordinates of the point of contact of the tangent (u, v, w) with (2) are

$$X = \frac{\Sigma(\mu_j x_i + \mu_i x_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j}{\Sigma(\mu_i + \mu_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j},$$

$$Y = \frac{\Sigma(\mu_j y_i + \mu_i y_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j}{\Sigma(\mu_i + \mu_j)\partial^2\Pi/\partial\alpha_i\partial\alpha_j}.$$

Let the tangent (u, v, w) coincide with the tangent through the two points P_p and P_q. Then

$$\frac{\partial^2\Pi}{\partial\alpha_i\partial\alpha_j} = 0 \qquad (\text{when } i \neq p,\ j \neq q),$$

since $\alpha_p = 0$, $\alpha_q = 0$ and α_p or α_q occurs in each of these second derivatives except $\partial^2\Pi/\partial\alpha_p\partial\alpha_q$. Hence the coordinates of the point of contact of (2) with P_pP_q are

$$X = \frac{\mu_q x_p + \mu_p x_q}{\mu_q + \mu_p}, \qquad Y = \frac{\mu_q y_p + \mu_p y_q}{\mu_q + \mu_p},$$

dividing the segment P_pP_q in the ratio $\mu_p : \mu_q$.

4. The foci of the curve $\varphi(u, v, w) = 0$ are the roots of the polynomial $\varphi(-1, -\sqrt{-1}, z)$* where $z \equiv x + y\sqrt{-1}$.

Thus in (2)

$$\alpha_i \equiv z - z_i, \qquad \Pi \equiv \prod_{i=1}^{n}(z - z_i).$$

Also

$$\varphi(-1, -\sqrt{-1}, z) \equiv \sum_{j=1}^{n}\mu_j \frac{\prod_{i=1}^{n}(z - z_i)}{z - z_i}.$$

But the derivative of $f(z)$ can be written

$$f'(z) = \prod_{i=1}^{n}(z - z_i)^{\mu_i - 1}\cdot \sum \mu_j \frac{\prod_{i=1}^{n}(z - z_i)}{z - z_j},$$

which establishes the theorem, § 2, as enunciated.

5. The present note is an abstract presenting the central theorem of a paper wherein the general curve $\varphi = 0$ of class

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* Salmon, Higher Plane Curves, 3d ed., § 141; also Emch, this BULLETIN, vol. 25, pp. 157–161.

$n-1$ is discussed and methods evolved for drawing it. We go no further here than to notice the interesting case when $n = 3$, of which the theorem of Lucas is a particular case.

6. When $n = 3$ the roots of the derivative of

$$f(z) \equiv \prod_{i=1}^{3} (z - z_i)^{\mu_i}$$

are the roots of $\prod_{1}^{3} (z - z_i)^{\mu_i - 1}$ and the foci of

$$\varphi \equiv \mu_1\alpha_2\alpha_3 + \mu_2\alpha_1\alpha_3 + \mu_3\alpha_1\alpha_2 = 0.$$

The curve is clearly of the second class and therefore a conic. The center z_c is given by

$$\frac{\partial}{\partial z}\Sigma\mu_j\frac{\Pi(z - z_i)}{z - z_j} = 0,$$

or

$$z_c = \frac{(\mu_2 + \mu_3)z_1 + (\mu_3 + \mu_1)z_2 + (\mu_1 + \mu_2)z_3}{2(\mu_1 + \mu_2 + \mu_3)}.$$

When the powers (μ) are positive the conic $\varphi = 0$ touches the sides of the triangle z_1, z_2, z_3 internally, and is therefore an ellipse inscribed in the triangle; the point of contact divides each side in the ratio of the powers of the vertices. If $\mu_1 = \mu_2 = \mu_3 = 1$, $f(z)$ becomes a cubic and the conic touches the sides of the triangle at their midpoints, which is Lucas's case.

When each power is not positive, it is only necessary to consider one as being negative, for changing the signs of all the powers does not effect $\varphi = 0$.

If $\mu_1 + \mu_2 + \mu_3 = 0$, the conic is a parabola, the center being at infinity. It follows from a well-known theorem, the focus of a parabola touching the three sides of a triangle lies on the circumcircle of the triangle, that in this case the roots of the derivative of $f(z)$ lie on a circle through the roots of $f(z)$.

A conic is clearly an ellipse or hyperbola according as it passes through a point between or not between two parallel tangents, respectively.

The tangents parallel to the sides of the triangle are easily found, since the center is known and the point of touch on a

side. Let the side z_1z_2 be the x-axis. The equation of the tangent parallel to this axis is

$$y = \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + \mu_3} y_3,$$

with similar equations of the tangents parallel to the other two sides. Let μ_3 be negative. The points of contact with the sides z_1z_3 and z_2z_3 lie between the x-axis and the parallel tangent or outside them according as

$$\mu_1 + \mu_2 + \mu_3 \lesseqgtr 0,$$

and the conic is accordingly an ellipse or hyperbola.

Hence if all the powers be positive $\varphi = 0$ is an ellipse. If one of the powers be negative it is an ellipse, parabola or hyperbola according as their sum is less than, equal to or greater than zero. If two of the powers be negative it is an ellipse, parabola or hyperbola according as their sum is greater than, equal to or less than zero. In particular if $\mu_1 + \mu_3 = 0$ the side z_1z_3 is an asymptote. If $\mu_1 = \mu_2 = - \mu_3$ the sides z_1z_3 and z_2z_3 are asymptotes. If in addition the triangle $z_1z_2z_3$ is isosceles having its vertex at z_3, the circle having z_3 for its center and passing through z_1 and z_2 passes through the foci of $\varphi = 0$, for the altitude and half base of this triangle are the major and minor axes respectively.

UNIVERSITY OF VIRGINIA,
May 1, 1919.

MORITZ CANTOR, THE HISTORIAN OF MATHEMATICS.

BY PROFESSOR FLORIAN CAJORI.

(Read before the San Francisco Section of the American Mathematical Society June 17, 1920.)

PROFESSOR Moritz Cantor died at Heidelberg on April 10, 1920, in his ninety-first year, three days after the death of his only son. Considering his own long career and that of many other mathematicians, he could well remark on his eightieth birthday, "Die Mathematik gehört nicht gerade zu den ungesunden Handwerken."* His literary activity ex-

* Felix Müller, Der Mathematische Sternenhimmel des Jahres 1811, Leipzig und Berlin, 1911, p. 7.

tended over a very long period, from 1851 to 1908. He has been known for many years as the leading writer of his time on the general history of mathematics. In the nineteenth century, his Vorlesungen über Geschichte der Mathematik came to be regarded as the fullest and the most reliable general source of historical information on our science. Before his time the outstanding historian of mathematics had been the Frenchman, J. E. Montucla, whose history appeared in 1758 in two volumes, and in 1799–1802 in four volumes (completed with the cooperation of the astronomer J. J. Lalande). Moritz Cantor is the Montucla of the nineteenth century.

Cantor was born in Mannheim on August 23, 1829. In 1848 he attended the University of Heidelberg and later studied at Göttingen under K. F. Gauss and W. Weber.* In the fall of 1851 he took his doctorate at Heidelberg, presenting a thesis "Ueber ein wenig gebrauchtes Koordinatensystem," which as yet disclosed no interest in that field of research which later he cultivated so assiduously. Curtze adds: "Before he came to devote himself wholly to an academic career—a state's examination he never took—he went to Berlin for study, where he attended especially the lectures of Lejeune-Dirichlet." We have seen nothing to indicate that Cantor had been in contact with, or had been influenced by, the Heidelberg privat-docent and Lyceum professor, Arthur Arneth, the author of the Geschichte der reinen Mathematik, Stuttgart, 1852.† Of three or more articles which Cantor prepared in the next few years, only one short paper of a dozen pages is historical, viz., "Ueber die Einführung unserer Ziffern," 1856, published in the *Zeitschrift für Mathematik und Physik*. What was the circumstance that induced him to choose the history of mathematics as his life work? The present writer visited him on May 2, 1915, and, in the course of conversation, asked him this very question. He replied that he prepared a paper on Ramus, Stifel and Cardan,‡ which he read at a scientific meeting at

* For facts of his early life we depend upon an article by Maximilian Curtze in *Bibliotheca Mathematica*, ser. 3, vol. 1, Leipzig, 1900, pp. 227–231.

† See M. Cantor's sketch of Arneth in the Allgemeine Deutsche Biographie, Leipzig, 1875.

‡ The article is "Petrus Ramus, Michael Stifel, Hieronymus Cardanus, drei mathematische Charakterbilder aus dem 16. Jahrhundert," *Zeitschrift für Mathematik und Physik*, vol. 2, 1857, pp. 353–367.

Bonn and which was so well received that he felt encouraged to continue in historical work. Thus, a little praise proved in his case a deciding factor. Perhaps this experience led him later in life to deal leniently with the shortcomings of young writers and to show generous appreciation of any merit he was able to discover. Soon after, Cantor went to Paris and there met Michel Chasles and Joseph Bertrand. Chasles had published in 1837 his Aperçu Historique and enjoyed wide celebrity as the historian of geometry. He paid much attention to the young German and secured the publication in the *Comptes Rendus* of a historical note prepared by the latter. Cantor prided himself on the command that he acquired of the French language. He told the writer that after delivering an address in French at the Mathematical Congress of 1900 in Paris, two Frenchmen came up and asked him to decide a bet: One Frenchman arguing that no foreigner could learn to speak French so well; the other claiming that Cantor was a German living in Germany.

It was in 1863 that Cantor's first important historical book appeared, his Mathematische Beiträge zum Kulturleben der Völker, which attracted wide attention, and was both praised and criticized. It championed an oft-repeated but seldom realized ideal in the writing of the history of our science, namely, the exhibition of the place of mathematics in the cultural life of a people and in the intercourse between nations. In this book, and in the first, and part of the second, volumes of the Vorlesungen, Cantor kept this goal in view. In the discussion of arithmetic and the history of our numerals, as given in his Beiträge, Cantor was influenced not only by the French writers M. Chasles, A. J. H. Vincent, and Henri Martin, but also by the early papers of the German G. Friedlein. Later Cantor was in frequent correspondence with the great French historian, Paul Tannery.

In 1864 Maximilian Curtze of the gymnasium of Thorn sought the assistance of Cantor in the interpretation of passages in a medieval manuscript. Thus began a friendship which continued unbroken until the death of Curtze in 1903.

Having been privat-docent at Heidelberg since 1853, Cantor advanced in 1863 to the position of professor extraordinary of mathematics. Probably about this time he began to give lectures on the history of mathematics. These lecture

courses extended sometimes over a period of three semesters.* According to Curtze, among those attending Cantor's lectures was Siegmund Günther, who is well-known as a historian of science.†

For the next ten or twelve years Cantor's publications were mainly historical. We make special mention of his Euklid und sein Jahrhundert (1867) and his Die Römischen Agrimensoren (1875). These served as preliminary researches for his great work, the Vorlesungen über Geschichte der Mathematik, the first volume of which appeared in 1880 when he was over fifty years of age. His previous studies, extending over a period of a quarter of a century, had been preparatory for this standard work. In entering upon this gigantic task he showed a predilection far remote from that of Paul Tannery. Cantor was writing a general history, while Tannery preferred to exercise his genius in elucidating special fields and special periods. A second edition of this first volume appeared in 1894, a third in 1907. Twelve years elapsed between the first and second volumes of the Vorlesungen. This second volume (1892) carried the history from the year 1200 to the year 1668, when Leibniz and Newton were just entering upon their great careers of discovery. A second edition of the second volume appeared in 1899. During the years 1894–1898 were published successive parts of the third volume which carries the history down to 1758, the time of the rise of Lagrange in the mathematical firmament. A second edition was brought out in 1901. These three great volumes enjoyed an enthusiastic reception. In 1898 Professor George A. Gibson of Glasgow only moderately expressed the feeling of his time when he said:‡ "That the labour involved in collecting material and in reducing it to shape would be great, Mr. Cantor doubtless knew well; but in all probability his most liberal estimate of the demands likely to be made upon his energies has been far exceeded; in any case, one can readily understand the feelings of satisfaction with which he writes the preface to the concluding volume. It hardly requires to be stated that this history is certain to remain for many years the standard work on the

* *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht* (Schotten), vol. 43, Leipzig und Berlin, 1912, p. 526.

† *Bibliotheca Mathematica*, ser. 3, vol. 1, p. 229.

‡ *Proceedings of the Edinburgh Mathematical Society*, vol. 17, 1898, p. 9.

subject with which it deals; in completeness, in accuracy, in clearness of arrangement, it stands unrivalled, and for the period which it covers is bound to be a permanent work of reference."

A great historian must be a judge and not an advocate. In this respect Cantor can be assigned a high place. In very few instances, if any, does national bias disclose itself. In this regard it would be difficult to name a historian of science who ranks above him.

At the time of the completion of the third volume, Cantor was sixty-nine years old. He knew that he was not equal to the task of preparing a fourth volume. On the other hand, the publishers and the mathematical public desired a continuation of the history. Finally, in 1904, at the Congress in Heidelberg, the plan was matured of bringing out a fourth volume on the cooperative plan. Nine men of different countries (V. Bobynin, A. v. Braunmühl, F. Cajori, S. Günther, V. Kommerell, G. Loria, E. Netto, G. Vivanti, C. R. Wallner) were selected to write certain parts under the direction of Cantor as editor-in-chief. Each collaborator was made responsible for his part. As far as possible each man was expected to follow the mode of presentation adopted in the previous volumes. The men were cautioned not to permit themselves to be dominated by preconceived ideas relating to the subjects or men treated in the history.* The fourth volume carried the history to 1799, the year of Gauss' doctor-dissertation.

It is the fate of a scientific man to be succeeded by others who carry his researches to a higher degree of perfection. Fraunhofer mapped the solar spectrum. His work was improved by E. Bequerel in France and J. W. Draper in America, both of whom used photography. Still greater perfection was introduced subsequently by A. Ångström, H. Rowland, and others. Moritz Cantor had a similar experience. The twentieth century set new standards. High as his Vorlesungen towered above other general histories of mathematics (such as those of J. E. Montucla, C. Bossut (1802), A. G. Kästner (1796–1800), J. H. M. Poppe (1828), A. Arneth (1852), F. Hoefer (1874), M. Marie (1883–1888)), it came to be criticized with great severity for lack of accuracy

* For further details see *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 13 (1904), pp. 475–478.

in minute details. It became evident that the erection of a great and faultless architectural structure must be preceded by the painstaking labors of the quarryman. Gustav Eneström, the editor of the *Bibliotheca Mathematica,* devoted much of his time for fifteen years to recording corrections to the first three volumes of Cantor's Vorlesungen. The Herculean efforts of Cantor and the extremely penetrating criticisms of Eneström clearly point out two lessons to scholars of today: (1) The need of a more accurate general history of mathematics, prepared on the scale of that of Cantor's Vorlesungen and embracing the historical researches of the last twenty years; (2) the impossibility of this task for any one man. A history of the desired size and accuracy can be secured only by the cooperative effort of many specialists. The mode of preparation of Cantor's fourth volume points the direction to future success.

The prodigious literary activity of Moritz Cantor can be seen also on examination of the volumes of the *Zeitschrift für Mathematik und Physik* for the years 1856–1898, and the volumes of Allgemeine Deutsche Biographie. The mere enumeration of the titles of his book reviews and of his articles fills twenty-six pages.*

As previously stated, the present writer on May 2, 1915, visited Cantor at Heidelberg, who was then in his eighty-sixth year. Almost complete blindness had compelled him to stop research. He said he could see the general outline of a person's body, but could not make out the features. He could move about in his house without assistance. His hearing was still good and his mind fairly clear. Neither of us touched upon war issues, except that he once referred to me as coming from a *nominally* neutral country. Later in the day I met his son and daughter, who were free in expressing to me the hope that the United States would soon come to observe *real* neutrality. Cantor made inquiries about certain American and English mathematicians and spoke of special historical researches then in progress in Germany. It was at his suggestion, he said, that K. Bopp undertook the study of Gregory St. Vincent.

* See M. Curtze "Verzeichnis der mathematischen Werke, Abhandlungen und Recensionen des Hofrat Professor Dr. Moritz Cantor," in Abhandlungen zur Geschichte der Mathematik, Neuntes Heft, Leipzig, 1899, pp. 625–650.

He spoke of Siegmund Günther as one of his pupils and of the late A. v. Braunmühl as an indirect pupil. The former, he said, had offered himself for service in the war, but was too old to fight. Cantor spoke in high praise of Hermann Hankel, and declared that his early death had been a great loss to mathematics; would that he had lived long enough to have completed his history of mathematics! Hankel, he said, was excellent not only in subject matter but also in the style of presentation. He spoke of his own labors as those of a hewer of timber who with a big axe and with powerful strokes roughly cut the timber to proper form and dimension, but left it for those who follow him to dress, polish and finish. When mention was made of Zeno's arguments on motion, Cantor referred to Paul Tannery's brilliant discussion of that subject. He spoke of Caspar Wessel's noted paper on imaginaries as having been completely neglected for about a century, notwithstanding the fact that Wessel's name as well as the title of his paper were given in so prominent a publication as Poggendorff's Handwörterbuch; H. G. Zeuthen discovered Wessel by accident. Cantor was much surprised that the name Oughtred was pronounced in England "ootred," and said he could not recall for the moment another English word in which "ough" was pronounced "oo." He referred to the mathematical precocity of his young grandson who spoke of the cellar as "minus 1. Stock." He said that in 1900 he discontinued the Abhandlungen zur Geschichte der Mathematik as a department of the *Zeitschrift für Mathematik und Physik;* nevertheless, only recently, a Vienna scientist had sent him a book for review therein.

He remembered Simon Newcomb at a Congress in Paris delivering an address in the French language when all Germans would have preferred to hear him speak English. At the Congress in Rome an Englishman arose, he said, to thank the King and the Italian mathematicians for the royal reception accorded to members of the Congress. The one who made reply for the Italian government declared his great admiration for the language of William Shakespeare, yet had to confess that his insufficient command of that language prevented him from following the Englishman's speech. A friend nudged the Italian speaker and called his attention to the fact that the Englishman's speech had been delivered in the Italian language. In speaking of Georg Cantor, who

had shortly before celebrated his seventieth birthday, he said that Georg Cantor's ancestors, as well as his own, came originally from Denmark, that, although he thought he and Georg Cantor were distantly related, the latter did not agree with him. After a dinner which was served in the garden in the open, the writer, having been told that it was Cantor's habit to lie down after the mid-day meal, departed.

UNIVERSITY OF CALIFORNIA.

SHORTER NOTICES.

Les Spectres numériques. By MICHEL PETROVITCH, with a preface by EMILE BOREL. Paris, Gauthier-Villars, 1919.

THE "spectral method" of Professor Petrovitch will be understood through an example taken from the last chapter of his monograph.

Let it be required to determine the coefficients in the development of $(1 + z)^6$. We have

$$(1.01)^6 = 1.061520150601,$$

and the binomial coefficients sought appear in the decimal above, some of the coefficients being separated by zeros.

It is clear that a similar method can be used to determine any finite set of positive integers, if the integers are the coefficients in the development of a known rational integral function, and that generalizations to infinite sequences of integers are possible.

Professor Petrovitch calls the decimal above a "spectrum" of the function $(1 + z)^6$, comparing the binomial coefficients to the colored portions of the spectrum, and the zeros to the dark portions. The book, up to the last chapter, is given up to a discussion of methods for associating "spectra" with different classes of functions.

In spite of the amiable preface of Borel, nothing will be found in the work to make one feel that the author has a point of view which is likely to lead to fruitful developments. Professor Petrovitch has certain speculations, but they are not of enough merit to warrant this address to the mathematical public.

Throughout the book are scattered references to results of contemporary analysts, and these contribute a certain amount of animation to the text, although the theorems quoted lead only to trivial conclusions in regard to the matter in hand. The style of the book is sufficiently elegant, and would be worthy of a more valuable scientific production.

J. F. RITT.

La Part des Croyants dans les Progrès de la Science au XIX^e Siècle. By ANTONIN EYMIEU. Première partie: Dans les sciences exactes. Troisième édition. Paris, Perrin et Cie., 1920. ii + 272 pp. Price 5 francs.

THIS little volume of entertaining but not particularly scientific summer reading is of interest to the mathematician chiefly because of the biographical material contained in the first chapter. It had its inception in connection with an anecdote which is often told of a verbal encounter in the Chamber of Deputies some thirty years ago. M. Charles Dupuy, speaking as rapporteur de l'Instruction publique, had spoken of the fatal anæmia of the facultés catholiques which, when they entered upon scientific studies, reached at last a stage where their faith called out, "Tu n'iras pas plus loin." The remark, so the official journal of the day records, was hailed with cries of approval from the Right, at which manifestation an opponent exclaimed, "As if there had never been any Christian scholars!" To this M. Dupuy replied, amid laughter, that it would be an interesting thing to see the list.

What M. Eymieu proposed for himself some five years ago, when the first edition appeared, was to meet the challenge and to show that it was those of religious faith who made the greatest contributions to the exact sciences in the nineteenth century. He admits, however, that it is an impossibility to prepare a complete catalogue of scientists and of their religious beliefs, and so he sets about to furnish a brief list, limited to the greatest contributors to mathematics, astronomy physics, and chemistry.

In mathematics M. Eymieu has selected the names of Gauss, Cauchy, Poincaré, Lagrange, Abel, Galois, Riemann, Weierstrass, and Hermite as representing the great research scholars—"les grand initiateurs," of whom, as he says, "de l'aveu de tous les bons juges, trois . . . dominent son

histoire," namely, Gauss, Cauchy, and Poincaré. It is of interest to observe that two out of these three who are said to have dominated the mathematical history of the century are French. Of the nine great leaders, four are also of the author's nationality, or five if we count Lagrange, who spent the latter part of his life in Paris and who was of French extraction; three are German, and one is a Norwegian. Out of the nine, the author proves to his satisfaction that five and probably six were religious men, that two (Lagrange and Galois) had no interest in religion, and that there remains a doubt as to the position of Poincaré. The two leading mathematical astronomers he takes to have been French, namely, Laplace and Leverrier, and each he claims to have been possessed of undoubted religious faith.

What is chiefly interesting in the work is that which the French call the "causerie" in which the author indulges. It is his gossip, his anecdotes about men, and his happy method of quoting interesting passages from speakers like Poincaré, Arago, and Bertrand that make the book readable. Such a human touch is seen in his quotation from that troublesome young Bolshevik, Galois, who was too advanced even for a France that claimed to stand for the most progressive thought of a century ago. The incident is well known, but is always interesting when one considers the sudden closing up of a life that was mentally brilliant and physically and morally a failure. The night before the duel in which he met his death he wrote to his republican patriots: "Je meurs victime d'une infâme coquette et de deux dupes de cette coquette. C'est dans un misérable cancan que s'eteint ma vie. . . . Adieu, j'avais bien de la vie pour le bien public."

M. Eymieu has been more successful in his selection of the nine great leaders, however open to criticism this may be, than in his choice of other great mathematicians who were distinctly religious in their faith. It is with some surprise that, among these "grands mathématiciens qui furent aussi de grands croyants," one finds that Boncompagni's name "leads all the rest." Boncompagni was a great man, and a great historian of mathematics, but he was not a great mathematician. The second name in the minor list is that of Joseph Bayma "que les journaux américains, en annonçant sa mort, proclamaient 'le géant des mathématiques,'"—after which statement it is quite unnecessary, at least for purposes of review, to examine the list any further.

The fact is that M. Eymieu has done his cause no good in the eyes of a scientific reader. His selections of men have not, in general, been made with care; indeed, they have not been made with ordinary knowledge. He has not scientifically gone to work to secure his information, as witness his uncertain results concerning the faith of Simon Newcomb. He has simply set about to support the belief of the uneducated or the half educated man of his own religious faith. It cannot be expected that he should have done for the dead what Professor Leuba did with respect to the religious beliefs of living scientists, but no one who has worked in the history of mathematics can fail to see that a much stronger case could have been made, and legitimately made, if the author had studied the problem with greater care.

It is evident to everyone that the most difficult thing to weigh in a scientific balance is the religious belief of mankind. The reasons are equally evident. One thing is clear, however,—that the study of the exact sciences no more tends to lessen this religious faith than the study of commerce, of civics, of sociology, or even of theology. The history of the exact sciences offers abundant illustrations of this fact, and evidence of a more convincing kind than that which M. Eymieu has adduced. Indeed, it would be a strange and inexplicable thing if scientific investigation should fail to show that mathematics, that branch of knowledge which is continually in touch with the infinite and is continually revealing the mysteries of the eternal, should fail to foster religious faith to an extent not reached by the other subjects of human study.

DAVID EUGENE SMITH.

The Early Mathematical Manuscripts of Leibniz. By J. M. CHILD. Chicago, 1920. iv + 238 pp.

THIS important work consists of translations of various Latin manuscripts of Leibniz found by Dr. C. I. Gerhardt in the Royal Library of Hanover about seventy-five years ago. These manuscripts were published by Dr. Gerhardt as parts of three works which he wrote on the origin of the differential and integral calculus,* and have long been known in their

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* Historia et Origo Calculi Differentialis, a G. G. Leibnizio conscripta, Hanover, 1846.

Die Entdeckung der Differentialrechnung durch Leibniz, Halle, 1848.

Die Geschichte der höheren Analysis; erste Abtheilung: Die Entdeckung der höheren Analysis, Halle, 1855.

Latin form to scholars who cared to consult these sources. There is no denying the fact, however, that the present generation of students in this country has not been trained to look upon Latin as a medium of intellectual exchange, and even in England it is undoubtedly true that the language has lost enough of its former prestige to make a translation of such material as the early manuscripts of Leibniz a great convenience to anyone who is interested in the subjects treated. For this reason there can be no question as to the value of this work and as to the renewed interest which it will awaken in the problem of the actual contribution of Leibniz to the revealing of the laws of the calculus and to the invention of a convenient symbolism.

The first document translated is the Historia et Origo, which Gerhardt published in 1846. This article is written in the third person, in the style which American readers have recently seen in the autobiography of Henry Adams, and was probably intended for anonymous publication. The story is here told, in popular fashion, of the steps which led Leibniz to his discovery. His disturbance over the publication of the Commercium Epistolicum in 1712, a publication which everyone who has examined it must admit is not a judicial document, seemed to him to require an answer. For this reason he proceeded to relate the incidents of his early publication, at the age of twenty, of the De Arte Combinatoria (1666); of his interest at the same time in general questions of analysis, including the theory of finite differences; of his visit to London at the age of twenty-six; of his acquaintance with Huyghens; and of the gradual development of his ideas of the calculus. He summarizes his case against the British school in the following words:

"Since therefore his opponents, neither from the Commercium Epistolicum that they have published, nor from any other source, brought forward the slightest bit of evidence whereby it might be established that his rival used the differential calculus before it was published by our friend;* therefore all accusations that were brought against him by these persons may be treated with contempt as beside the question."

The second body of translated material consists of several manuscripts of the period 1673–1680. These relate to various topics bearing upon the calculus. For example, on November

* That is, by Leibniz himself.

1, 1675, he treats of moments about axes, giving none of the new symbols, *omn.* being still used instead of $\int$. Ten days later (November 11), the symbol $\int$ is used, x/d and dx are stated to be interchangeable, differentials of the second order are rejected, the problem is solved of finding a curve such that the rectangle contained by the subnormal and ordinate is constant, and Barrow's form of the differential triangle is used. Ten days later still, November 21, he speaks of a new kind of trigonometry of indivisibles, in which reference the editor finds the influence of both Pascal and Barrow. In June, 1676, he writes that "the true general method of tangents is by means of differences." In the following month (July, 1676) he writes, partly in Latin and partly in French, on the inverse method of tangents, and in November of the same year he sets forth a number of the basic laws for differentiation and integration. In July, 1677, however, he shows no evidence of ability to differentiate logarithmic, exponential, or trigonometric functions, but by 1680 he seems to have been in possession of the general theory which he published in the *Acta Eruditorum* in 1684.

This brief summary will serve to give some idea of the material awaiting the further study of scholars;—further study, because the world will not be inclined to accept as satisfactory the study given by either Dr. Gerhardt or Mr. Child. Dr. Gerhardt was a careful student, but he is shown by Mr. Child not to have been altogether judicial in his statements. As to the editor of the present work, two questions will occur to any reader who carefully examines his contribution. In the first place, has he approached the subject with the unprejudiced mind of a searcher after truth? Mr. Child has himself answered this question (page 229):

"I therefore set out with the determination to break down, if possible, the credibility of Leibniz as a witness in his own defense, when it came to unimportant details; then to show that he had opportunities for obtaining everything necessary to the development of the calculus, that he could not be expected to supply for himself by original work, without having need to know anything of the work of Newton; then to show that these sources of information were set out in a form far more suitable to the requirements of Leibniz than the work of Newton; finally, to clinch the matter, that the analogy of Leibniz's work was so close to these sources, that it was idle to suppose that he made use of any other sources."

It is impossible to read this confession of prejudice without being conscious of a slight feeling of, or akin to, that of amusement in Mr. Child's strictures upon Gerhardt:

"Never surely did any man have such a glorious opportunity as Gerhardt, in the whole history of scientific controversies; surely there never was an advocate who left himself so open to the attacks of the opponents."

It is to be regretted that Mr. Child repeats so often his ideas of the dependence of Leibniz upon Barrow. This repetition is partly due no doubt to the fact that the book is made up of essays that appeared from time to time in *The Monist*, little effort having been made to unify the presentation of the matter when combined in book form. It is also to be regretted that a more restrained style was not possible, since such a style would have carried much greater conviction than the one adopted. Probably the best approach to a perfect description of the working of a human mind in the reaching of a mathematical discovery is that given by Lord Moulton in his address at the Napier celebration at Edinburgh in 1914. There the trained intellect of a mathematician and an eminent jurist concentrated on giving a clear analysis of the development of a great idea, and the result of the analysis was a masterpiece,—delightful in style, free from any apparent bias, and convincing in its conclusions.

It is too much to expect, however, that we can all be Lord Moultons. Perhaps it is more human to find ourselves influenced by Carlyle as we see him in his thoroughly biased essay on Cromwell. But the reader will soon find that with all of Mr. Child's evident scholarship and painstaking research, and in spite of all our indebtedness to him for his excellent translation and the information which he often gives us, in the matter of style he is not particularly fortunate. Illustrations like the following will tend to convince the reader, however generous may be his inclinations, that the work was not carefully revised before publication:

"Does not this silence on the part of Tschirnhaus, the personal friend of Leibniz, rather tend to make Leibniz's plea, that his opponents had had the shrewdness to wait till Tschirnhaus, among others, was dead, recoil on his own head, in that he has done the very same thing?" (Page 29.)

"The work of Descartes, looked at at about the same time as Clavius, that is, while he was still a youth, 'seemed to be more intricate.'" (Page 37.)

"This, without either proof or figure, is a hopeless muddle. . . . Goodness knows what the use of it was supposed to be in this form!" (Page 61.)

"Neither Gerhardt nor Weissenborn tried to get to the bottom of these manuscripts, being content with simply 'skimming the cream.'" (Page 74.)

"Thus what is generally considered to be a muddle turns out to be quite correct. The muddle is not with Leibniz, it is with the transcriber." (Page 81.)

"This is of course nonsense." (Page 97.)

"I cannot get out of my head the suggestion that . . . ." (Page 110.)

"Is Leibniz trying to draw a red herring across the trail, the real trail that leads to Barrow's *a* and *e*?" (Page 128.)

Unfortunately, there are a large number of similar instances that will strike the reader's attention as he studies the pages of the book.

It is a rather low type of criticism that looks only for the misprints and inconsequential slips of the pen in the work of an author. When Mr. Childs remarks that "there is of course the usual misprint (in Gerhardt's work) that one is becoming accustomed to," he tempts the reader, however, to recall various instances of a similar kind in the work under review. Without wishing to call attention to these misprints in detail, the point may be illustrated by the cases of 15 for 16 (page 31), the period for a comma on page 74 (line 5), the absence of an interrogation point after the question in the note on page 101, and the date 1874 for 1674 as that of the second edition of Barrow (page 13).

That the book is a valuable contribution to the history of mathematics, however, is evident to anyone who gives its pages even a casual reading.

David Eugene Smith.

An Enquiry Concerning the Principles of Natural Knowledge. By A. N. Whitehead. Cambridge University Press, 1919. xii + 200 pp.

The aim of this work is to illustrate the principles of natural knowledge by an examination of the data and the experiential laws fundamental for physical science. The modern theory of relativity has opened the possibility of a new answer to the question as to how space is rooted in experience and has

brought to light a new world of thought as to the relations of space and time to the ultimate data of perceptual knowledge. "The present work is largely concerned with providing a physical basis for the more modern views which have thus emerged. The whole investigation is based on the principle that the scientific concepts of space and time are the first outcome of the simplest generalizations from experience, and that they are not to be looked for at the tail end of a welter of differential equations."

Three main streams of thought—the scientific, the mathematical, and the philosophical—are relevant to the theme of the enquiry. About half the book is given to parts I and II on the traditions of science and the data of science respectively. In part III on the method of extensive abstraction we have a philosophical and postulational treatment of the space-time manifold; and this is employed in part IV to yield a theory of objects. The fundamental assumption elaborated "is that the ultimate facts of nature, in terms of which all physical and biological explanation must be expressed, are events connected by their spatio-temporal relations, and that these relations are in the main reducible to the property of events that they contain (or extend over) other events which are parts of them."

R. D. CARMICHAEL.

Differential Equations. By H. BATEMAN. Longmans, Green and Company, London and New York, 1918. xii + 306 pp.

"THE subject of Differential Equations has grown so rapidly in recent years that it is difficult to do justice to all branches of the subject in a single volume." So reads the opening sentence of the preface to the book under review. The statement is literally true; but it is nevertheless of such form that its connotation may be misleading to the learner. One who is not acquainted with the subject of differential equations may justly conclude from this sentence that it is possible, though indeed difficult, to do justice to all branches of the subject in a single volume. And yet it is probable that no one who knows the field would be willing to maintain such a judgment. It is in fact true that it would be difficult to do justice to all branches of the subject in ten volumes of the size of the one under consideration.

In the second sentence of his preface the author indicates

the purpose of his book in the following language: "In writing this book I have endeavored to supply some elementary material suitable for the needs of students who are studying the subject for the first time, and also some more advanced work which may be useful to men who are interested more in physical mathematics than in the developments of differential geometry and the theory of functions."

From this it will be seen that the author in his single exposition seeks to serve two rather different purposes. In so far as these purposes diverge there is an evident possibility that the provision for one will interfere with the best provision for the other. It seems to the reviewer, in fact, that the more advanced matter has been provided in such way as to interfere with the usefulness of the book to students who are studying the subject for the first time. There does not appear to be a sufficient indication (even with the help of the preface) as to what parts of the book are best suited to the needs of the beginner. It is not clearly apparent, indeed, that the author himself had in mind a definite separation of material for serving the two purposes named. Moreover, the exposition from time to time assumes on the part of the reader a range of mathematical knowledge which is not the common possession of students pursuing the subject for the first time. A similar statement may also be made relative to those who are interested primarily in physical mathematics.

These considerations may lead to the fear that the usefulness of the book has not been increased by the attempt to serve simultaneously two distinct purposes which are not so intimately bound together as to justify this plan of treatment without a more distinct separation and indication of parts to serve these purposes.

Occasionally the rhetorical relations within a sentence might be improved. Greater care in proofreading would have resulted in the avoidance of several minor blemishes. But so far as the reviewer has observed these defects are not likely to cause the reader serious inconvenience.

Having said these things, let us turn now to matters which it is more pleasant to dwell upon. Even a rapid examination of the contents of the volume will bring out the fact that it contains a considerable amount of useful material not easily accessible elsewhere, and indeed some new material in Chapters VII and VIII.

Chapter II (pages 17–59) on integrating factors contains a rather wide range of useful and interesting matter, a part of which one would hardly expect to find under the title given. Further elementary methods of solution are treated in Chapter III (pages 60–86) under the heading transformations. In Chapter V (pages 103–115) we have a brief treatment of differential equations with particular solutions of a specified type. Some rather interesting results are obtained by starting from this elementary point of view, results which it would probably be more difficult to secure by any other method. They have to do especially with the algebraic integrals of certain important equations. Chapters VI to XI bear the following titles: partial differential equations, total differential equations, partial differential equations of the second order, integration in series, the solution of linear differential equations by means of definite integrals, the mechanical integration of differential equations.

Chapter VIII (pages 169–222) on partial differential equaions of the second order will be found particularly rich in material with an appeal to those who are interested primarily in physical mathematics. Here one finds a considerable treatment of the equation of wave propagation in three dimensions, of the Maxwell equations, of the electron equations, of Laplace's equation, and of the equation of the conduction of heat.

The chapter on solutions by means of definite integrals contains a useful summary of material, a part of which is not so conveniently found elsewhere so far as the reviewer is aware.

Those who give instruction to elementary classes in differential equations will find in this book of Bateman's a useful source of supplementary material of the nature both of additional topics for special report by members of the class and of problems to furnish a variation from those in the basic text. Some of the problems here included are rather unusual in character (as for instance 2 and 6 and 17 of the miscellaneous examples) and hence are of value to the instructor who wishes to introduce greater variety.

R. D. CARMICHAEL.

NOTES.

THE concluding (June) number of the *Annals of Mathematics* contains the following papers: "Algebraic surfaces, their cycles and integrals, by S. LEFSCHETZ; "The potential of ring-shaped disks," by E. P. ADAMS; "Total differentiability. A correction," by E. J. TOWNSEND; "Existence theorem for the non-self-adjoint linear system of the second order," by H. J. ETTLINGER; "Motion in a resisting medium," by J. K. WHITTEMORE; "Continuous matrices, algebraic correspondences, and closure," by A. A. BENNETT; "Urn schemata as a basis for the development of correlation theory," by H. L. RIETZ; "On pseudo-resolvents of linear integral equations in general analysis," by T. H. HILDEBRANDT.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1920–1921:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Principles of celestial mechanics; figures of equilibrium of rotating fluid masses; problems of cosmogony, three hours.—By Professor L. DONATI: Elementary and general relativity, three hours.—By Professor F. ENRIQUES: Elementary mathematics in the light of higher concepts. History of the ideas; criticism; problems, three hours.—By Professor S. PINCHERLE: Analytic functions; integral analytic functions; ordinary differential equations from Lie's standpoint, three hours.

UNIVERSITY OF CATANIA.—By Professor M. CIPOLLA: Theory of groups of finite order with applications, four hours.—By Professor G. SCORZA: Enumerative geometry; theory of invariants, five hours.—By Professor ——: Mathematical physics, three hours.

UNIVERSITY OF GENOA.—By Professor G. LORIA: Infinitesimal geometry of common space and of hyperspaces, three hours.—By Professor C. SEVERINI: Theory of analytic functions, four hours.—By Professor O. TEDONE: Elementary theories of electricity and magnetism, three hours.

UNIVERSITY OF MESSINA.—By Professor P. CALAPSO: Functions of a complex variable and elliptic functions, four

hours.—By Professor G. Giambelli: Singularities of algebraic curves; introduction to the geometry on an algebraic curve, four hours.—By Professor O. Lazzarino: Vector operations and functions; fields of force; principles of electrostatics and of the statics of elastic bodies, four hours.

University of Naples.—By Professor F. Amodeo: History of mathematics: the century of Galileo Galilei, three hours.—By Professor A. Del Re: Analytic theory of heat, four and one-half hours.—By Professor R. Marcolongo: Electrodynamics; theory of Lorentz; optical applications, three hours.—By Professor D. Montesano: Geometry of straight lines and of conics in the ordinary spaces, six hours.—By Professor E. Pascal: Abelian integrals and functions, three hours.

University of Padua.—By Professor U. Amaldi: Contact transformations and canonical systems, four hours.—By Professor F. d'Arcais: Monogenic functions; gamma and elliptic functions, four hours.—By Professor P. Gazzaniga: Theory of numbers, three hours.—By Professor G. Ricci: Methods of absolute differential calculus, with applications to the theory of elasticity, four hours.—By Professor F. Severi: Calculus of probabilities, four hours.—By Professor A. Tonolo: Partial differential equations of the first order, three hours.

University of Palermo.—By Professor G. Bagnera: Complex variables and integral analytic functions; elliptic functions, three hours.—By Professor M. De Franchis: A study of the real parts of algebraic entities, three hours.—By Professor M. Gebbia: Vector fields, electro- and magnetostatics, four and one-half hours.—By Professor A. Signorini: Theory of relativity, three hours.—By Professor V. Strazzeri: Riemann surfaces; algebraic curves; abelian integrals, three hours.

University of Pavia.—By Professor L. Berzolari: Abelian integrals, three hours.—By Professor U. Cisotti: Relativity; Einstein's mechanics, three hours.—By Professor F. Gerbaldi: Functions of a complex variable and elliptic functions, three hours.—By Professor F. Sibirani: Plane and twisted curves; principles of the theory of surfaces and con-

gruences of rays, three hours.—By Professor G. VIVANTI: Theory of functions of a real variable; Lebesgue integrals, three hours.

UNIVERSITY OF PISA.—By Professor G. ARMELLINI: Celestial mechanics, three hours.—By Professor E. BERTINI: Projective geometry of hyperspaces, three hours.—By Professor L. BIANCHI: Functions of a complex variable; algebraic numbers and analytical arithmetic, three hours.—By Professor G. A. MAGGI: Analytical dynamics; equilibrium and vibration of elastic bodies; elastic theory of light, four and one-half hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometric applications of the calculus, three hours.—By Professor E. BOMPIANI: Continuous groups of transformations, three hours.—By Professor F. CANTELLI: Mathematical statistics, three hours; Actuarial mathematics, three hours.—By Professor G. CASTELNUOVO: Complex variables and algebraic functions, three hours.—By Professor U. CRUDELI: Introduction to advanced theories of electricity and magnetism, three hours.—By Professor T. LEVI-CIVITA: Absolute differential calculus with applications, three hours.—By Professor A. PERNA: Complements of mathematical analysis, three hours.—By Professor L. SILLA: Differential equations of dynamics, three hours.—By Professor V. VOLTERRA: General relativity, three hours; Differential equations of mathematical physics, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Hydrodynamics, three hours.—By Professor G. FUBINI: Differential geometry and continuous groups, with special reference to the groups of motions and of conformal and projective transformations.—By Professor C. SEGRE: Geometry of differential equations, three hours.—By Professor C. SOMIGLIANA: Theory of oscillations and electromagnetic optics, three hours.—By Professor G. TOGLIATTI: Projective differential geometry, three hours.

The following courses are announced in American universities for the academic year 1920–1921, in addition to those listed in previous numbers of the BULLETIN.

UNIVERSITY OF ILLINOIS.—By Professor E. J. TOWNSEND: Complex variables, three hours; Differential equations, three hours.—By Professor G. A. MILLER: Theory of groups (introductory course), three hours; Theory of equations, three hours (first term).—By Professor J. B. SHAW: Vector methods, three hours.—By Professor A. B. COBLE; Algebraic and abelian functions, three hours.—By Professor R. D. CARMICHAEL: Linear difference equations, three hours.—By Professor A. EMCH: Geometric transformations, three hours (first term); Geometry in a complex field, three hours (second term).—By Professor A. R. CRATHORNE: Theory of statistics, three hours.—By Professor G. E. WAHLIN: Theory of numbers, three hours.—By Professor H. BLUMBERG: Theory of aggregates, three hours (second term).—By Professor E. B. LYTLE: History of mathematics, two hours (second term).

YALE UNIVERSITY.—By Professor E. W. BROWN: Differential equations, three hours (second term); Advanced mechanics, three hours.—By Professor J. PIERPONT: Theory of functions of real variables, two hours; Theory of functions of a complex variable, three hours.—By Professor W. R. LONGLEY: Theory of differential equations, two hours.—By Professor E. J. MILES: Advanced calculus, three hours (first term); Differential geometry, two hours.—By Professor J. I. TRACEY: Modern and differential geometry, three hours; Analytic geometry, two hours.—By Professor J. K. WHITTEMORE: Advanced differential geometry, two hours.—By Dr. W. L. CRUM: Mechanics, three hours.—By Mr. J. S. MIKESH: History of mathematics, two hours.—By Dr. J. M. STETSON: Higher algebra, two hours.

THE class of sciences of the Belgian academy announces the following subjects for prize memoirs for 1921: (1) a contribution to the study of the properties of analytic functions that do not take certain values in a given domain; (2) a contribution to the study of birational transformation in a space of more than two dimensions.

THE adjudicators of the Hopkins prize of the Cambridge philosophical society have made the following awards: for the period 1903–06, to Dr. W. BURNSIDE, of Pembroke College,

for investigations in mathematical science; for the period 1906–09, to Professor G. H. BRYAN, of Peterhouse, for investigations in mathematical physics, including aerodynamic stability; and for the period 1909–12, to Mr. T. R. WILSON, of Sidney Sussex College, for investigations in physics, including the paths of radio-active particles.

PROFESSOR H. HAPPEL, of the University of Tübingen, has been appointed to a professorship at the Breslau technical school.

AT the University of Münster, Professor R. COURANT, of the University of Göttingen, has been appointed professor of mathematics. Associate professor H. KONEN has been promoted to a full professorship of theoretical physics.

AT the University of Kiel, Dr. O. TOEPLITZ has been promoted to a full professorship of mathematics, and Dr. E. MADELUNG to a full professorship of theoretical physics. Professor H. JUNG has resigned, to accept an appointment as professor of mathematics at the University of Halle.

THE Braunschweig technical school has conferred the honorary degree of doctor of engineering on Professor H. LORENZ, of the Danzig technical school.

AT the University of Rostock, associate professor R. WEBER, of the department of applied mathematics and physics, has been promoted to an honorary professorship; Dr. O. HAUPT, of the Karlsruhe technical school, has been appointed at a professorship; Professor O. STAUDE has received the honorary degree of doctor of engineering from the technical school of Darmstadt.

ASSOCIATE professor R. WEITZENBÖCK, of the German technical school at Prague, has been promoted to a full professorship of mathematics.

THE following persons have recently been admitted as privat-docents in German technical schools and universities: Dr. A. BARUCH, in mathematics, in the department of mines at the Berlin technical school; Dr. J. DEUXES, in mathematics, at the University of Göttingen; Dr. A. LANDE, in theoretical physics, at the University of Frankfurt a. M.

Dr. A. Chatelet has been appointed professor of mathematics at the University of Lille.

Dr. Bouligand has been appointed maître de conférences in mathematics at the University of Rennes.

At Cambridge University, Mr. J. E. Littlewood, of Trinity College, has been appointed Cayley lecturer in mathematics, and Mr. J. H. Grace, of Peterhouse, has been reappointed University lecturer in mathematics.

Professor L. E. Dickson, of the University of Chicago, has been elected correspondent of the Paris academy of sciences, in the section of geometry, as successor to Professor E. Cosserat, elected non-resident member.

At the University of Texas, assistant professor R. L. Moore, of the University of Pennsylvania, has been appointed associate professor of pure mathematics. Associate professor Jessie M. Jacobs, of Rockland College, has been appointed instructor in pure mathematics and assistant professor J. N. Michie, of the Texas agricultural and mechanical college, has been appointed adjunct professor of applied mathematics.

Yale University has conferred the honorary degree of master of arts on Professor H. E. Hawkes, dean of Columbia College.

Dr. Daniel Buchanan, professor of astronomy and mathematics at Queen's University, Kingston, Ontario, has been appointed professor of mathematics and head of the department at the University of British Columbia. At Queen's University, assistant professor C. F. Gummer has been promoted to an associate professorship, and Dr. Norman Miller to an assistant professorship of mathematics; Mr. A. Woods has resigned, to accept a lecturership at Western University, London, Ontario.

Professor S. Lefschetz, of the University of Kansas, has been granted leave of absence for the year 1920–1921. He will spend the first term at the University of Paris, and the second at Padua and Bologna.

Assistant professor E. G. Bill, of Dartmouth College, has been promoted to a full professorship of mathematics.

Dr. J. W. Campbell, of the State University of Iowa, has been appointed assistant professor of mathematics at the University of Alberta.

Miss Gertrude Smith, of Vassar College, has been promoted to an assistant professorship of mathematics.

Dr. J. R. Kline, of the University of Illinois, has been appointed assistant professor of mathematics at the University of Pennsylvania.

Dr. L. R. Ford, of Harvard University, has been appointed assistant professor of mathematics at Rice Institute.

At Elmira College, Miss Mary Suffa, of Beloit College, has been appointed head of the department of mathematics, and Miss Francis W. Wright, of Brown University, has been appointed instructor.

Assistant professor A. C. Maddox, of the Oklahoma Agricultural and Mechanical College, has been appointed professor of mathematics at the Louisiana State Normal School at Natchitoches.

Mr. F. H. Murray and Dr. J. L. Walsh have been appointed Sheldon Fellows by Harvard University, and expect to spend the coming year studying in Paris.

Professor Ida Barney, of Meredith College, has been appointed assistant professor of mathematics at Smith College.

Assistant professor Louisa M. Webster, of Hunter College, has been promoted to an associate professorship of mathematics.

Dr. C. A. Laisant died May 5, 1920, at the age of seventy-nine years. He was founder of the *Intermédiare des Mathé-*

maticiens, one of the founders of *l'Enseignement Mathématique*, and an editor of the *Nouvelles Annales de Mathématiques*.

Miss Mary A. Colpitts, instructor in mathematics at the University of Wisconsin, died July 11, 1920.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Ahrens (W.). Mathematiker-Anekdoten. 2te, stark veränderte Auflage. (Mathematisch-physikalische Bibliothek, Nr. 18.) Leipzig, Teubner, 1920. 42 pp. M. 1.40

Barthel (E.). Polargeometrie. Berlin, Leonhard Simion Nachfolger, 1919. M. 4.50

Chini (M.). Esercizi di calcolo infinitesimale. 3a edizione. Livorno, Giusti, 1920. 8vo. 10 + 300 pp. L. 8.50

Encyklopädie der mathematischen Wissenschaften. Band II 3, Heft 3: L. Lichtenstein, Neuere Entwicklung der Potentialtheorie. Konforme Abbildung. Leipzig, Teubner, 1919. M. 7.60

Fricke (R.). See Perry (J.).

Lichtenstein (L.). See Encyklopädie.

Luckey (P.). Einführung in die Nomographie. 2ter Teil: Die Zeichnung als Rechenmaschine. Leipzig, Teubner, 1920. M. 1.40

Meissner (O.). Wahrscheinlichkeitsrechnung. 1tes Bändchen: Grundlehren. 2tes Bändchen: Anwendungen. Leipzig, Teubner, 1919. M. 1.00 + 1.00

Muir (T.). The theory of determinants in the historical order of development. Volume 3: The period 1861 to 1880. London, Macmillan, 1920. 26 + 503 pp. 35s.

Pasch (M.). Mathematik und Logik. 4 Abhandlungen. Leipzig, 1919. M. 2.00

Perry (J.). Höhere Mathematik für Ingenieure. Autorisierte deutsche Bearbeitung von R. Fricke und F. Süchting. 3te Auflage. Leipzig, Teubner, 1919. M. 20.00

Piaggio (H. T. H.). An elementary treatise on differential equations and their applications. London, Bell, 1920. 12s.

Pincherle (S.). Lezioni di calcolo infinitesimale dettate nella R. Università di Bologna e redatte per uso degli studenti. 2a edizione riveduta. Bologna, Zanichelli, 1920. 8vo. 8 + 785 pp. L. 40.00

Riemann (B.). See Weber (H.).

SMITH (W. W.). A theory of the mechanism of survival: the fourth dimension and its applications. London, Kegan Paul and Company, 1920. 196 pp. 5s.

STONEY (J.). Calculus for engineering students. New York, Pitman, 1920. 12mo. 140 pp. $1.35

SÜCHTING (F.). See PERRY (J.).

WEBER (H.). Die partiellen Differentialgleichungen der mathematischen Physik. Nach Riemanns Vorlesungen bearbeitet. 1ter Band. 6te Auflage. Braunschweig, 1919. M. 14.00

II. ELEMENTARY MATHEMATICS.

CRANTZ (P.). Arithmetik und Algebra zum Selbstunterricht. 2ter Teil: Gleichungen. 5te Auflage. Leipzig, 1919. M. 1.75

CRESPI (B. I.) e SOLUSTRI (A.). Esercizi e problemi di aritmetica, geometria, algebra, ad uso delle scuole medie inferiori. Roma, tip. del Genio civile, 1920. 8vo. 192 pp. L. 4.25

FROUMENTY (M.). See PHILIPPE (P.).

GERLACH (A.). Von schönen Rechenstunden. 4te vermehrte Auflage. Leipzig, Quelle und Meyer, 1919. Geb. M. 8.00

PANIZZA (F.). Aritmetica pratica. 4a edizione, riveduta e corretta. (Manuali Hoepli.) Milano, Hoepli, 1920. 24mo. 9 + 191 pp. L. 3.00

PHILIPPE (P.) et FROUMENTY (M.). Cours de géométrie. Tome 1. 2e édition. Paris, Dunod, 1920. 12mo. 8 + 272 pp. Fr. 9.75

SOLUSTRI (A.). See CRESPI (B. I.).

III. APPLIED MATHEMATICS.

ANGERSBACH (A.). Das Relativitätsprinzip leichtfasslich entwickelt. (Mathematisch-physikalische Bibliothek, Nr. 39.) Leipzig, Teubner, 1920. 57 pp.

BAIRSTOW (L.). Applied aerodynamics. London, Longmans, 1920. 12 + 566 pp. 32s.

BRENDEL (M.). See KLEIN (F.).

BROSE (H. L.). See SCHLICK (M.).

BROWN (E. W.). Tables of the motion of the moon. Prepared with the assistance of H. B. Hedrick. Six sections. New Haven, Yale University Press (printed by the Cambridge University Press), 1919. 13 + 140 + 39 + 223 + 99 + 56 + 102 pp. $20.00

DEPREZ (M.) et SOUBRIER (M.). Les lois fondamentales de l'électrotechnique. Paris, Dunod, 1920. 8vo. 758 pp. Cartonné. Fr. 34.50

DUPUIS (A.) et LOMBARD (J.). Cours de dessin industriel. Tome 3. 2e édition, nouveau tirage. Paris, Dunod, 1919. Fr. 9.00

EINSTEIN (A.). Ueber die spezielle und die allgemeine Relativitätstheorie. 4te Auflage. Braunschweig, 1919. M. 2.80

—— See LORENTZ (H. A.).

EWING (J. A.). Thermodynamics for engineers. Cambridge, University Press, 1920. 30s.

GAUSS (C. F.). See KLEIN (F.).

HEDRICK (H. B.). See BROWN (E. W.).

HENSELING (R.). Kleine Sternkunde. Stuttgart, Kosmos, Franckh, 1919. 109 pp. Geh. M. 3.60

HIRSCHAUER (L.). L'aviation de transport. L'évolution de la construction de 1907 à 1919 et la réalisation des avions de transport. L'utilisation économique des appareils. Paris, Dunod, 1919. 233 pp. + 3 cartes. Fr. 33.00

KLEIN (F.), BRENDEL (M.) und SCHLESINGER (L.). Materalien für eine wissenschaftliche Biographie von Gauss. Heft 7: Ueber die astronomischen Arbeiten von Gauss. 1ter Abschnitt: Theoretische Astronomie, von M. Brendel. Leipzig, Teubner, 1919. M. 6.00

LINDEMANN (F. A.). See SCHLICK (M.).

LOMBARD (J.). See DUPUIS (A.).

LORENTZ (H. A.), EINSTEIN (A.) und MINKOWSKI (H.). Das Relativitätsprinzip. 3te, verbesserte Auflage. Leipzig, Teubner, 1920. 146 pp.

MINKOWSKI (H.). See LORENTZ (H. A.).

NIEMÖLLER (M.). Zur Bewegung eines Punktes auf Rotationsflächen bei Wirkung eines auf der Rotationsachse gelegenen Anziehungszentrums. (Dissertation.) Halle, 1919.

RANELLETTI (C.). Elementi di geometria descrittiva, ad uso dei rr. istituti tecnici. 2a edizione. (Manuali Hoepli.) Milano, Hoepli, 1920. 24mo. 12 + 197 pp. con 2 tavole. L. 4.50

SCHEFFERS (G.). Lehrbuch der Mathematik für Studierende der Naturwissenschaften und der Technik. 4te verbesserte Auflage. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1919. M. 42.00

SCHLESINGER (L.). See KLEIN (F.).

SCHLICK (M.). Space and time in contemporary physics. Rendered into English by H. L. Brose. Introduction by F. A. Lindemann. Oxford, University Press, 1920. 6s. 6d.

SOUBRIER (M.). See DEPREZ (M.).

THOMSON (G. P.). Applied aerodynamics. London, Hodder and Stoughton, 1920. 20 + 292 pp. 42s.

VALENTINER (S.). Die Grundlagen der Quantentheorie in elementarer Darstellung. 2te, erweiterte Auflage. Braunschweig, 1919. M. 3.60

THE TWENTY-SEVENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-seventh summer meeting and ninth colloquium of the Society were held at the University of Chicago on September 7–11, following the summer meeting of the Mathematical Association of America, which was held on Monday, September 6. The sessions of Tuesday and of Wednesday forenoon were devoted to the reading of papers; the colloquium opened on Wednesday afternoon.

The members of the Society and their guests found excellent arrangements for their entertainment at Hitchcock and Beecher Halls, and at the Quadrangle Club. The Club was very graciously put at the disposal of the members for meals and for use as a social center. On Friday afternoon visits were made under the guidance of Professor Slaught to Ida Noyes Hall and to Harper Memorial Library.

At the joint dinner of the two organizations, one hundred and sixteen persons were present. Professor Slaught, retiring president of the Mathematical Association, presided at the after-dinner speaking, which was participated in by several of those present, to the edification of all the listeners.

The meetings of the Society were attended by more than one hundred and twenty persons, among whom were the following ninety-nine members of the Society:

Professor R. C. Archibald, Professor G. N. Armstrong, Professor I. A. Barnett, Professor Suzan R. Benedict, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor G. A. Bliss, Professor R. L. Borger, Professor J. W. Bradshaw, Professor W. C. Brenke, Professor W. H. Bussey, Professor W. D. Cairns, Professor J. W. Campbell, Professor A. L. Candy, Professor E. W. Chittenden, Professor C. E. Comstock, Professor A. R. Crathorne, Dr. G. H. Cresse, Professor D. R. Curtiss, Professor H. H. Dalaker, Professor S. C. Davisson, Professor E. L. Dodd, Professor L. W. Dowling, Professor Arnold Dresden, Professor Otto Dunkel, Professor M. D. Earle, Professor Arnold Emch, Professor G. C. Evans, Professor H. S. Everett, Professor Peter Field, Professor B. F. Finkel, Professor L. R. Ford, Professor W. B. Ford, Professor M. G. Gaba, Professor W. H. Garrett, Professor

D. C. Gillespie, Professor R. E. Gilman, Dr. T. H. Gronwall, Professor C. F. Gummer, Professor W. L. Hart, Professor M. W. Haskell, Professor Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. A. Hopkins, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. D. C. Kazarinoff, Professor S. D. Killam, Professor H. W. Kuhn, Professor Gillie A. Larew, Professor Flora E. LeStourgeon, Professor A. C. Lunn, Professor E. B. Lytle, Professor R. B. McClenon, Professor J. V. McKelvey, Professor T. E. Mason, Professor Helen A. Merrill, Professor Bessie I. Miller, Professor W. L. Miser, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Professor A. L. Nelson, Professor Anna H. Palmié, Professor Anna J. Pell, Dr. T. A. Pierce, Professor A. D. Pitcher, Professor S. E. Rasor, Professor R. G. D. Richardson, Professor H. L. Rietz, Professor Maria M. Roberts, Professor W. H. Roever, Professor Oscar Schmiedel, Dr. Caroline E. Seely, Professor E. W. Sheldon, Dr. W. G. Simon, Professor E. B. Skinner, Professor H. E. Slaught, Mr. H. L. Smith, Professor E. B. Stouffer, Professor C. E. Stromquist, Professor K. D. Swartzel, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Oswald Veblen, Dr. J. H. Weaver, Professor W. P. Webber, Mr. F. M. Weida, Professor Mary E. Wells, Professor W. D. A. Westfall, Professor Marion B. White, Professor C. E. Wilder, Professor F. B. Wiley, Professor C. H. Yeaton, Professor J. W. Young, Professor J. W. A. Young.

Vice-President R. G. D. Richardson presided at the meetings of Tuesday and Wednesday forenoons; Professor Haskell presided on Tuesday afternoon.

Upon the recommendation of the Council the Society adopted the following amendments to the By-Laws:

1. By-Law II, section 2, is amended by changing the words "five dollars" to "six dollars," so that the amended section reads "The annual dues shall be six dollars payable on the first of January, etc." Adopted unanimously.

2. By-Law II, section 3, is amended by changing the words "fifty dollars" to "seventy-five dollars," so that the amended section reads "On the payment of seventy-five dollars in one sum, any member of at least four years standing and not in arrears of dues may become a life member and shall thereafter be exempt from all annual dues." Adopted by a vote of 47 to 2.

3. By-Law IX, section 2 is amended by the addition of the sentence: "One member of the Committee of Publication shall be designated by the Council as Editor of the BULLETIN." Adopted unanimously.

The Council announced the election of the following persons to membership in the Society: Dr. R. F. Borden, Brown University; Dr. Tso Chiang, Nan Kai College, Tientsin, China; Professor H. M. Dadourian, Trinity College, Hartford, Conn.; Dr. Jesse Douglas, Columbia University; Mr. Philip Franklin, Princeton University; Dr. C. F. Green, University of Illinois; Captain R. S. Hoar, Ordnance School, Aberdeen, Md.; Dr. Jessie M. Jacobs, University of Texas; Dr. E. L. Post, Princeton University; Professor C. D. Rice, University of Texas; Mr. L. G. Simon, New York City; Professor J. E. Stocker, Lehigh University; Dr. Tsao-Shing Yang, Syracuse University. Twenty-two applications for membership in the Society were received.

The following resolution was adopted unanimously: "The American Mathematical Society expresses to the department of mathematics of the University of Chicago its heartiest appreciation of the arrangements they have made for the twenty-seventh summer meeting of the Society and its gratitude for their hospitality."

Professor Hedrick spoke on behalf of the Committee on increase of membership and of sales of the Society's publications. A considerable number of new subscriptions to the *Transactions* were secured subsequently.

Professor Bliss then offered the following resolution, which was adopted unanimously: "The Society recommends for favorable consideration by the Council applications for membership from advanced students and others interested in mathematics, whether engaged in teaching or not, when properly proposed by members of the Society."

The following papers were read at this meeting:

(1) Professor ARNOLD EMCH: "On the projective generation of cyclides."

(2) Dr. J. H. WEAVER: "A generalization of the strophoid."

(3) Professor C. F. GUMMER: "On the relative distribution of the real roots of two real polynomials."

(4) Professor A. A. BENNETT: "The polyadic expansion of a number."

(5) Dr. J. L. WALSH: "On the location of the roots of the jacobian of two binary forms."

(6) Dr. J. L. Walsh: "On the transformation of convex point sets."

(7) Professor W. B. Ford: "On Kakeya's minimum area problem."

(8) Professor T. H. Hildebrandt: "On completely continuous linear transformations."

(9) Professor Anna J. Pell: "Integral equations in which the kernel is quadratic in the parameter."

(10) Professor Olive C. Hazlett: "Annihilators of modular invariants."

(11) Professors Virgil Snyder and F. R. Sharpe: "Construction of multiple correspondences between two algebraic curves."

(12) Professor Dunham Jackson: "Note on a method of proof in the theory of Fourier's series."

(13) Professor J. W. Campbell: "On the drift of spinning projectiles."

(14) Professor W. L. Hart: "Functions of infinitely many variables in Hilbert space."

(15) Professor D. C. Gillespie: "A property of continuity."

(16) Professor L. A. Hopkins: "Periodic orbits of type 2/1."

(17) Professor Dunham Jackson: "Note on the median of a set of numbers."

(18) Professor C. N. Moore: "An application to Weierstrass's function of the generalized derivative of type $(C\,1)$."

(19) Dr. L. R. Ford: "A method of graduating curves."

(20) Professor E. W. Chittenden: "Note on a generalization of a theorem of Baire."

(21) Professor E. W. Chittenden: "On classes of functions defined in terms of relatively uniform convergence."

(22) Professor E. W. Chittenden: "On the relation between the Hilbert space and the calcul fonctionnel of Fréchet."

(23) Dr. J. L. Walsh: "A generalization of the Fourier cosine series."

(24) Professor Dunham Jackson: "Note on a class of polynomials of approximation."

(25) Professor G. A. Miller: "Reciprocal subgroups of an abelian group."

(26) Professors E. R. Hedrick, W. D. A. Westfall and Louis Ingold: "Characteristic lines of transformations."

(27) Professor W. L. HART: "Pseudo-differentiation of a summable function with respect to a parameter."

(28) Professor EDWARD KASNER: "Five notes on Einstein's theory of gravitation."

(29) Professor DUNHAM JACKSON: "On the convergence of certain trigonometric approximations."

(30) Professor DUNHAM JACKSON: "Note on the Picard method of successive approximations."

(31) Professor OLIVE C. HAZLETT: "A symbolic notation in the theory of formal modular invariants."

(32) Dr. T. H. GRONWALL: "On the Fourier coefficients of a continuous function."

(33) Dr. T. H. GRONWALL: "A sequence of polynomials connected with the nth roots of unity."

(34) Dr. T. H. GRONWALL: "Upper bounds for the coefficients in conformal mapping."

The papers of Professors Snyder and Sharpe, Dr. Walsh, Professor Miller, Professor Kasner, Professor Chittenden's first paper, Professor Jackson's fifth paper and Dr. Gronwall's first and second papers were read by title. Abstracts of the papers numbered in accordance with the above list of titles are given below.

1. Professor Emch shows how cyclides may be treated from the projective point of view. In the first place it is proved that every cyclide may be generated in an infinite number of ways by two projective pencils of spheres (of which one may degenerate into a pencil of planes). By this method of generation many of the well-known properties of cyclides may easily be derived. It also leads to a generalized cyclide which may be called a quintic cyclide, whose definition is contained in the following theorem:

The tangent planes of a quadric cone and an independent projective pencil of spheres generate a quintic cyclide with a finite circle as a double curve. This quintic degenerates into an ordinary general cyclide when the radical plane of the pencil of spheres coincides with the projectively corresponding tangent plane of the cone.

When α, β, γ are linear quantics representing planes, P and Q quadratic quantics representing spheres, and λ denotes a parameter, the cone of class 2 and the projective pencil of spheres have the form

$$\alpha\lambda^2 + \beta\lambda + \gamma = 0, \quad P - \lambda Q = 0,$$

and the quintic cyclide is

$$\alpha P^2 + \beta PQ + \gamma Q^2 = 0.$$

2. Among the generalizations of the strophoid given by Loria in his treatise Algebraische und transcendente ebene Kurven, the one by W. W. Johnson is as follows: Let there be two fixed points A and B and let two lines l_1 and l_2 make angles φ_1 and φ_2 with the line AB such that

$$m\varphi_1 \pm n\varphi_2 = \alpha \quad (\alpha = \text{constant}).$$

Then if l_1 and l_2 intersect in P, the locus of P is a strophoid. Since α is a constant there is associated with the strophoid a a circle passing through A and B. Dr. Weaver has developed strophoids having associated with them the three conic sections in the same relation that the circle is associated with the strophoid of Johnson, and has proved a number of theorems on concurrence of lines and collinearity of points connected with the curves.

3. The roots of a real polynomial which lie within a given interval on the real axis are divided into groups by the roots of a second real polynomial. Professor Gummer shows how this distribution may be determined by a rational process. A generating function $\Sigma_r P_r (1 + t)^{k-r}(1 - t)^r$ is formed such that the coefficients of descending powers of t are the numbers in the successive groups from left to right. Two methods are given for the determination of the P's, corresponding to the theorems of Sturm and Hermite for a single polynomial. Sylvester had given a method for the reduced arrangement in which pairs of roots of either polynomial occurring consecutively were omitted.

4. In this paper Professor Bennett discusses the array consisting of all the p-adic expansions of a given integer, as p ranges through the sequence of primes. For a given p, the p-adic coefficients constitute a row of the array and for a given n, as p varies, the coefficients of p^n constitute a column. For every integer, positive or negative, there is an array, or polyadic expansion, while a polyadic expansion does not in general represent an actual integer. Sections of this array form the natural instruments in the study of numerical modular domains with a composite modulus. Subtraction and division are considered.

5. An annular region is defined as a closed region of the plane bounded by two non-intersecting circles. Dr. Walsh proves that if three annular regions are respectively the envelopes of three points z_1, z_2, z_3, then the envelope of the point z_4 defined by the real constant cross ratio

$$\lambda = (z_1, z_2, z_3, z_4)$$

is also an annular region. This result is applied in giving some geometric results concerning the location of the roots of the jacobian of two binary forms.

If f_1 and f_2 are binary forms of respective degrees p_1 and p_2, then the roots of the jacobian of f_1 and f_2 are the real foci of a certain curve of class $p_1 + p_2 - 1$ which touches all the lines formed by joining the pairs of roots of f_1 and f_2 in all possible ways. The curve has various other interesting properties.

6. The chief purpose of Dr. Walsh's note is to prove that a necessary and sufficient condition that a one-to-one point transformation of the plane or of space transform every convex point set into a convex point set is that it be a collineation.

7. Kakeya's problem in its simplest form is as follows: How should a line segment AB be turned end for end in a plane so as to sweep out a minimum area? Professor Ford shows that there are an infinite number of methods of attaining such a minimum area if it be assumed that, as the segment moves, the area generated nowhere returns into itself; but if it be assumed that it returns into itself exactly once in one or more regions, that is, if duplication be allowed, then there is no minimum area attainable by motions confined to the finite plane.

8. In a paper entitled "Lineare Funktionalgleichungen"* F. Riesz has discussed the linear integral equation in an elegant manner, which on account of its fundamental character seems destined to point the way to new and more general results in allied fields. In the first part of Professor Hildebrandt's paper a simple general basis is provided in which the methods and results of Riesz are valid. The second part of the paper takes up a more detailed study of the inversion of the com-

* *Acta Mathematica*, vol. 41 (1916), pp. 71–96.

pletely continuous transformation which leads to a generalization of the notion of the solution of adjoint homogenous integral equations, and of pseudo-resolvents.

9. Mrs. Pell considers the questions of the existence of solutions of linear integral equations in which the kernel has the form $\lambda K(x, s) + \lambda^2 L(x, s)$ and K and L satisfy certain conditions, and the expansion of arbitrary functions in terms of the solutions.

10. In any theory of invariants, differential operators which annihilate invariants are useful in the computation of invariants and covariants, and are sometimes important in the development of the general theory. Professor Hazlett's paper determines such annihilators for modular invariants. These operators are in some ways analogous to the well known Aronhold operators in the theory of classical invariants, but are very much more complicated, as might be expected. They are, in fact, of the general type anticipated by Professor Dickson in a paper published in 1907.

11. It is well known (Hurwitz, *Mathematische Annalen*, volume 28) that algebraic correspondences between two algebraic curves exist which require two auxiliary equations for their definition. The condition that two equations shall be required was found by Castelnuovo (*Rendiconti dei Lincei*, 1906), but no illustrations have been given. The example given by Amodeo (*Annali*, series 2, volume 20), cited by Castelnuovo, of the intersection of two ruled surfaces in general position can be defined by one equation. The paper by Professors Snyder and Sharpe discusses this example, and gives various methods of constructing correspondences that require two equations for their definition. The paper will appear in the *Transactions*.

12. Professor Jackson's first paper will appear in a later number of the BULLETIN.

13. Professor Campbell treats the problem of drift for the case in which initially the rotation of the projectile is entirely about the axis. He obtains a formula which exhibits qualitatively the well-known characteristics of the phenomenon. An application to the British Mark VI gives results which are quantitatively consistent with observed values.

14. In considering functions of infinitely many real variables $(x_1, x_2, x_3, \ldots)$, the case which is of most interest as viewed from the standpoint of the theory of integral equations and of more general functional equations is that in which it is supposed that $\Sigma_{i=1}^{\infty} x_i^2$ converges. In the present paper, Professor Hart assumes that the points $\xi = (x_1, x_2, \cdots)$ considered satisfy this condition.

A function $f(\xi)$ is said to be completely continuous at a point $\xi = (a_1, a_2, \cdots)$ if the second of the following equations holds whenever the first does:

$$\lim_{n=\infty} \sum_{j=1}^{\infty} (x_{jn} - a_j)^2 = 0; \tag{1}$$

$$\lim_{n=\infty} f(x_{1n}, x_{2n}, \cdots) = f(a_1, a_2, \cdots). \tag{2}$$

Professor Hart first proves certain general theorems, regarding completely continuous functions $f(\xi)$, including a mean value theorem which gives, as a corollary, an expression for the differential of $f(\xi)$. There is then considered the proof of the existence of a continuous solution of the infinite system of equations

$$f_i(t, x_1, x_2, \cdots) = 0 \qquad (i = 1, 2, \cdots), \tag{3}$$

and the existence of a solution of the system of differential equations

$$\frac{dx_i}{dt} = g_i(t, x_1, x_2, \cdots) \qquad (i = 1, 2, \cdots). \tag{4}$$

In (3) and (4) the f_i and g_i are supposed to be completely continuous in their arguments.

15. A continuous function $f(x)$ has the property that between x_1 and x_2 there is at least one value of x for which $f(x)$ has any prescribed value between $f(x_1)$ and $f(x_2)$. This property, common to continuous functions, is possessed by some discontinuous functions. Professor Gillespie's note is concerned with functions which have this property. Conditions which are sufficient to insure their continuity are developed; the character of the discontinuities that such a function may have is shown; and a function, having this property and having its set of points of continuity and its set of points of discontinuity each everywhere dense, is constructed.

16. By combining the analytic processes of Poincaré with mechanical quadrature, Professor Hopkins has obtained a family of periodic orbits in the restricted problem of three bodies, which has significance in the study of the gap in the asteroids where the period would be one half the period of Jupiter, and in the consideration of Cassini's gap in the rings of Saturn. These orbits constitute an exceptional case in certain work of F. R. Moulton and are related to results of Poincaré, E. W. Brown and G. W. Hill.

17. Professor Jackson's second paper will appear in a later number of the BULLETIN.

18. In this paper Professor Moore shows that under certain conditions Weierstrass's function, $f(x) = \Sigma a^n \cos b^n \pi x$, whose derivative in the ordinary sense fails to exist, has a generalized derivative of type $(C1)$. The definition of such generalized derivatives has been given in two papers previously presented to the Society. (Cf. this BULLETIN, volume 25 (1919), pages 249 and 257.)

19. In the smoothing of irregular curves based on original data a satisfactory combination of smoothness and faithfulness to the data is desired. Professor Ford considers the case of a function $f(x)$, given for equidistant values of the argument $x = 0, 1, \cdots, m$, and determines the graduated function $y(x)$ by minimizing the sum

$$\sum_{x=0}^{m} \{M(x)[y(x) - f(x)]^2 + [\Delta^n y(x)]^2\},$$

where the weight $M(x)$ is positive, and where n is the order of differences it is desired to make as small as possible. This leads to reasonably simple numerical methods, and, if $M(x)$ is constant, the solution has the property, desirable in certain cases of graduation, that its moments, up to the $(n - 1)$th, about any point are the same as the corresponding moments of the ungraduated function.

20. Professor Chittenden's first paper appeared in full in the October number of the BULLETIN.

21. Denote by B_0 the class of all continuous functions defined on a limited perfect subset P of n-space; by B_α the class

of all functions which do not belong to any class of order less than α and are limits of sequences of functions of class less than α. The theory of the ordinal series of classes B_α has been developed by Baire, Lebesgue and Vallée-Poussin. In terms of the relatively uniform convergence of E. H. Moore, Professor Chittenden has previously defined the corresponding series of classes R_α and now reports on the results of further investigations.

A set E is of type (α, α) at most if its characteristic function is of class B_α at most. Denote by A_α the class of all functions of class B_α such that for every pair of numbers a, b the set $E = [a < f < b]$ is of type (α, α) at most; and by F_α the set of all functions of class B_α which assume a limited number of distinct values. Then $F_\alpha < A_\alpha$ and we have $F_{\alpha+1} < R_{\alpha+1} \leqq A_{\alpha+1} + B_\alpha$, while if α is of the second kind, then $F_\alpha \leqq R_\alpha \leqq A_\alpha < B_\alpha$. Necessary and sufficient conditions that a function be of class R_α are found; but, except in the case $\alpha \leqq 1$, it has not been determined whether or not A_α contains functions not in R_α. The classes F_α and A_α are investigated, with results of interest in the general theory of the series of classes B_α.

22. It is well known that the points of the Hilbert space Ω of infinitely many dimensions can be placed in one-to-one correspondence with the set Ω_1 of all summable functions on an interval I, if we agree to identify functions of Ω_1 which differ only on a set of points of measure zero. Fréchet has called attention to the fact that the subset φ of Ω which corresponds to the set of all continuous functions on I is not closed under the usual definition of distance for the Hilbert space and suggests the desirability of a definition of distance for that space relative to which the set φ will be closed.

Employing results of Féjer and the theory of convergence in mean, Professor Chittenden defines distance for the Hilbert space in terms of the coordinates of a point and trigonometric functions so that limit in the Hilbert space becomes equivalent to uniform convergence on I, excepting a set of points of measure zero. When the functions considered are continuous this implies the continuity of the limit function and leads to the desired closure of the set φ.

The paper will be published in the *Palermo Rendiconti*.

23. Dr. Walsh proves by use of the theory of infinitely many variables that if we consider the set of functions

$$\frac{1}{\pi}\cos\lambda_0 x,\quad \frac{2}{\pi}\cos\lambda_1 x,\quad \frac{2}{\pi}\cos\lambda_2 x,\quad \frac{2}{\pi}\cos\lambda_3 x,\ \cdots \qquad (0 \leqq x \leqq \pi),$$

where

$$\lambda_0^2 + 4(\lambda_1 - 1)^2 + 4(\lambda_2 - 2)^2 + 4(\lambda_3 - 3)^2 + \cdots < \frac{1}{\pi}$$

and where

$$\sum_{n=1}^{\infty} n^2 |\lambda_n - n|$$

converges, then there exists another set of functions $\{v_n(x)\}$ biorthogonal to that set. Moreover, for any function $f(x)$ integrable in the sense of Lebesgue and with an integrable square, the two series

$$\tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \cdots,\quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx,$$

$$\frac{b_0}{\pi}\cos\lambda_0 x + \frac{2b_1}{\pi}\cos\lambda_1 x + \frac{2b_2}{\pi}\cos\lambda_2 x + \cdots,$$

$$b_n = \int_0^{\pi} f(x)v_n(x)dx,$$

have essentially the same convergence properties throughout the interval $0 \leqq x \leqq \pi$.

24. In a paper recently presented to the Society (see this BULLETIN, June, 1920, page 391) it was shown that if $f(x)$ is a given continuous function in the interval $a \leqq x \leqq b$, n a given positive integer, and m a given real number greater than 1, there exists one and just one polynomial $\varphi(x)$, of degree n or lower, which reduces to a minimum the value of the integral

$$\int_a^b |f(x) - \varphi(x)|^m dx.$$

The purpose of Professor Jackson's third paper is to establish the truth of the same proposition in the case that $m = 1$. The existence of at least one minimizing polynomial is proved in essentially the same way as before, the details of the argument being somewhat simpler in the present case. The proof of uniqueness is considerably less direct, and appears to involve

elementary considerations of the measure of point sets. A notable difference between the two cases is that for $m > 1$ the remainder corresponding to the polynomial of best approximation must change sign at least $n + 1$ times in the interval, unless it vanishes identically, while for $m = 1$ the assertion is merely that the remainder must change sign at least $n + 1$ times, or else vanish throughout a finite number or an enumerable infinity of intervals contained in (a, b), and having at least a certain specified aggregate length.

25. Any two subgroups of the group G which have the property that the product of their orders is equal to the order of G have been called reciprocal subgroups of G. In the present paper Professor Miller confines his attention to reciprocal subgroups of abelian groups. Among the theorems proved are the following: If two reciprocal subgroups of any abelian group have only identity in common, the number of the conjugates of the one under the group of isomorphisms is equal to the number of the conjugates of the other under this group. Whenever an abelian group of order p^m, p being a prime number, contains subgroups of the same order but of different types, then the number of the subgroups of one and of only one of these types is of the form $1 + kp$. The number of the subgroups of each of the other types is divisible by p. A necessary and sufficient condition that the number of the subgroups of a given type contained in such a group is a power of p is that the number of its independent generators of each order increased by the number of the larger independent generators in the set is equal to the number of the independent generators of G whose orders are not less than this order.

26. In this paper Professors Hedrick, Westfall, and Ingold discuss the properties of a double orthogonal set of lines defined for any transformation that is not conformal by the property that their directions at each point are the directions of maximum and minimum stretching due to the transformation. These lines were originally defined by Tissot by a different property. It is shown that there always exists a transformation which corresponds to any preassigned set of such characteristic lines, and that the rate of stretching along one family may also be preassigned. Other properties are found and special cases are discussed.

27. Consider a function $u(s, t)$, where $a \leqq s \leqq b$ and where t assumes all values in a measurable set E. Let us suppose that, for every value of s, $u(s, t)$ becomes a function of t which, together with its square $u^2(s, t)$, is summable in the Lebesgue sense in E. Professor Hart defines $u_s(s_0, t)$ as the pseudo-derivative function of $u(s, t)$ with respect to s at the point s_0 if it satisfies the condition

$$\lim_{\Delta s=0}\int_E\left[\frac{u(s_0+\Delta s, t)-u(s_0, t)}{\Delta s}-u_s(s_0, t)\right]^2 dt = 0.$$

If $u_s(s_0, t)$ exists, it is unique except for its values at a set of points t of measure zero. If $u_s(s, t)$ exists and if $g(t)$ and $(g(t))^2$ are summable, the function

$$h(s) = \int_E u(s, t)g(t)dt$$

has a derivative dh/ds. It is found that the Fourier constants $[x_1(s), x_2(s), \cdots]$ and $[y_1(s), y_2(s), \cdots]$ of $u(s, t)$ and $u_s(s, t)$, respectively, relative to a complete, unitary orthogonal system of functions $[g_1(t), g_2(t), \cdots]$, satisfy the relations

$$dx_i(s)/ds = y_i(s) \qquad (i = 1, 2, \cdots).$$

Under suitable conditions regarding the summability of the function $u_s(s, t)$ in the rectangle $a \leqq s \leqq b$, $c \leqq t \leqq d$, there is given a generalization of the mean value theorem for the function $u(s, t)$. After a few auxiliary results are established, the solution of a certain type of pseudo-differential functional equations is obtained by means of certain theorems regarding differential equations in infinitely many variables.

28. Professor Kasner's notes all relate to four-dimensional riemannian manifolds $ds^2 = \Sigma g_{ij}dx_idx_j$ obeying Einstein's ten gravitational equations $R_{ij} = 0$, where the g's are the ten potentials and the equations denote the vanishing of the so-called contracted Riemann-Christoffel tensor (which might appropriately be called the Einstein tensor). The main results follow:

(1) It is known that if the paths of particles *and* light pulses can be regarded (in some coordinate system) as straight lines, then the ds^2 is necessarily equivalent to the euclidean form. The same result is here shown to follow if the hypothesis is limited to particles alone, or to light pulses alone.

(2) Can two Einstein manifolds ever have the same light paths? This is shown to be impossible, at least in the case of approximately euclidean manifolds. Application to the one-body problem (solar gravitation): the Mercury effect could be predicted from the observed light deflection.

(3) Approximately euclidean solutions which can be expressed as the sum of four squares; in particular the quasi-conformal type $A(dx_1^2 + dx_2^2 + dx_3^2) + Bdx_4^2$ (which includes the solar field) in explicit form.

(4) If the coefficients of the four squares are functions of a single variable, the only exact (finite) solutions are certain exponentials or powers.

(5) Discussion of the possible Einstein manifolds which can be regarded as immersed in a flat space of 5 or 6 dimensions. The solar field appears in the latter case.

29. This paper is concerned with the trigonometric sums $T_{mn}(x)$ which furnish the closest approximation to a given continuous periodic function $f(x)$, in the sense of the integral of the mth power of the absolute value of the error, for prescribed values of the exponent m and the order n of the sum. The question at issue is the convergence of $T_{mn}(x)$ to the value $f(x)$, when m is held fast ($m \geqq 1$) and n is allowed to become infinite. Professor Jackson shows that a sufficient condition for uniform convergence is that $\lim_{\delta=0} \omega(\delta)/\sqrt[m]{\delta} = 0$, where $\omega(\delta)$ is the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. For $m = 1$, it is sufficient that $f(x)$ have a continuous derivative. The proof makes use of Bernstein's theorem on the derivative of a trigonometric sum. As the condition obtained is less general, even for large values of m, than the well-known Lipschitz-Dini condition in the case of Fourier's series, $m = 2$, the present results may be regarded as preliminary.

30. In presenting the Picard existence proof for a differential equation $dy/dx = f(x, y)$, it is customary to assume that f is continuous, and satisfies a Lipschitz condition in y, throughout a certain rectangle in the xy-plane, and then to show that the successive approximations converge to a solution of the differential equation throughout a sufficiently restricted interval of values of x. Professor Jackson points out that by extending the definition of f outside the rectangle, so that the continuity conditions are maintained, but otherwise

arbitrarily, the process can be made to converge throughout the entire range of values of x originally considered. A function is obtained which satisfies the *original* differential equation as long as the x and y *of the solution* remain in the rectangle, whatever the behavior of the approximating functions may be. This remark is rather obvious, but it is conspicuously omitted from standard treatments of the subject. It applies equally well to systems of n differential equations in n unknown functions, and to regions that are not rectangular.

31. Professor Hazlett's second paper considers a symbolic notation for the modular invariants of a binary form with respect to the general Galois field $GF[p^n]$ of order p^n. In the classic theory of algebraic invariants, a convenient symbolic notation is obtained by expressing the binary form f of order n symbolically as a perfect nth power, $f = \alpha_x^n = \beta_x^n = \gamma_x^n = \cdots$, where $\alpha_x = \alpha_1 x_1 + \alpha_2 x_2$, $\beta_x = \beta_1 x_1 + \beta_2 x_2$, etc. Such a notation is not, however, practicable in the theory of modular invariants. If we express f as the product of n linear factors which are symbolically distinct, $f = \alpha_x \beta_x \cdots$, we have a symbolic notation by the aid of which we can write down all modular invariants of f, both formal and otherwise. In fact every formal modular invariant can be expressed as a polynomial in a finite number of symbolic expressions which behave like invariants. This theorem is illustrated for the binary cubic modulo 2 and the binary quadratic modulo 3. There is a systematic method by which any formal modular invariant for these two special cases can be expressed in symbolic form, and the writer suspects that such is true in general.

32. Given any real and single valued function $\varphi(x)$ which tends to infinity with x, Dr. Gronwall shows how to construct a function $f(\theta)$ continuous for $0 \leqq \theta \leqq 2\pi$ and such that its Fourier coefficients a_n, b_n (which have the well-known property that $\Sigma(a_n^2 + b_n^2)$ converges) make the series

$$\Sigma(a_n^2 + b_n^2)\varphi\left(\frac{1}{a_n^2 + b_n^2}\right)$$

divergent.

33. Dr. Gronwall considers a polynomial $F(z)$ of degree $n - 1$ which does not exceed unity in absolute value at the nth roots of unity, so that $|F(z)| \leqq 1$ for $z = 1, \epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$

where $\epsilon = e^{2\pi i/n}$. It is shown that on the unit circle $|z| = 1$ we have

$$(1) \qquad |F(z)| < \frac{1}{n}\sum_{\nu=0}^{n-1}\frac{1}{\sin\frac{2\nu+1}{2n}\pi}$$

except when $F(z)$ has the form $e^{\alpha i}f(\epsilon^{-k}z)$, where α is real, k an integer and

$$(2) \qquad f(z) = \sum_{\nu=0}^{n-1}\frac{(\epsilon^{-\frac{1}{2}}z)^\nu}{n\sin\frac{2\nu+1}{2n}\pi},$$

in which case the upper bound of $|F(z)|$ is reached at $z = \epsilon^{\frac{1}{2}+k}$. The polynomial $f(z)$ has all its zeros on the unit circle, one in each of the intervals from ϵ to ϵ^2, ϵ^2 to ϵ^3, $\cdots$, ϵ^{n-1} to 1. The asymptotic value of the upper bound in (1) is

$$\frac{2}{\pi}\left(\log n + C + \log\frac{2}{\pi}\right) + \sigma(1)$$

where C is Euler's constant, and $\sigma(1)$ tends to zero as n increases indefinitely.

34. Dr. Gronwall shows that when $w = z + a_2z^2 + \cdots a_nz^n + \cdots$ maps the circle $|z| < 1$ conformally on a simply connected and nowhere overlapping region in the w-plane, then $|a_n| \leqq n$ for $n = 2, 3, \cdots$. When $|a_n| = n$ for any particular n, then also $|a_n| = n$ for every n, and the function w reduces to the one which gives extreme values to the distortion.

ARNOLD DRESDEN,
Acting Secretary.

THE CHICAGO COLLOQUIUM.

THE ninth colloquium of the American Mathematical Society was held in connection with its twenty-seventh summer meeting at the University of Chicago, September 8–11, 1920. At the annual meeting of 1917, the Council, on the invitation of the Department of Mathematics of the University of Chicago, appointed a committee, consisting of Professors E. H. Moore, E. W. Brown, Max Mason, H. S. White, and the Secretary, to arrange for a summer meeting and

colloquium to be held at Chicago in 1919. At the annual meeting of 1918 the Council authorized the postponement of the Chicago meeting until the summer of 1920. The courses of lectures were announced in the preliminary circular of May, 1920, and printed syllabi were distributed at the meeting. The colloquium opened Wednesday afternoon, September 8, and continued until Saturday noon; three lectures were delivered on Thursday, and Friday, and two on each of the other days. At the close of each lecture opportunity for discussion was given, on points of that or previous lectures.

The following ninety persons were in attendance, a number considerably exceeding that of any previous colloquium.

Mr. E. S. Akeley, Professor R. C. Archibald, Professor G. N. Armstrong, Professor I. A. Barnett, Professor S. R. Benedict, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor G. A. Bliss, Professor R. L. Borger, Professor J. W. Bradshaw, Professor W. C. Brenke, Professor W. H. Bussey, Professor W. D. Cairns, Professor J. W. Campbell, Mr. F. E. Carr, Mr. W. E. Cederberg, Professor E. W. Chittenden, Professor A. R. Crathorne, Dr. G. H. Cresse, Professor D. R. Curtiss, Professor E. L. Dodd, Professor L. W. Dowling, Professor Arnold Dresden, Professor Otto Dunkel, Mr. J. D. Eshleman, Professor G. C. Evans, Professor H. S. Everett, Dr. L. R. Ford, Professor W. B. Ford, Professor M. G. Gaba, Professor D. C. Gillespie, Professor R. E. Gilman, Dr. T. H. Gronwall, Professor C. F. Gummer, Professor W. L. Hart, Professor M. W. Haskell, Professor Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. A. Hopkins, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. D. C. Kazarinoff, Miss Claribel Kendall, Professor S. D. Killam, Professor H. W. Kuhn, Professor Gillie A. Larew, Professor Flora E. LeStourgeon, Mrs. Mayme I. Logsdon, Professor A. C. Lunn, Mr. C. C. MacDuffee, Professor Helen A. Merrill, Professor Bessie I. Miller, Professor W. L. Miser, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Professor A. L. Nelson, Mr. H. L. Olson, Miss Eleanor Pairman, Professor Anna H. Palmié, Professor Anna J. Pell, Dr. T. A. Pierce, Professor A. D. Pitcher, Professor S. E. Rasor, Professor R. G. D. Richardson, Professor H. L. Rietz, Professor W. H. Roever, Miss I. M. Schottenfels, Dr. Caroline Seely, Professor E. W. Sheldon, Dr. W. G. Simon, Professor E. B. Skinner,

Professor H. E. Slaught, Professor H. L. Smith, Professor C. E. Stromquist, Professor K. D. Swartzel, Dr. Bird M. Turner, Professor A. L. Underhill, Professor E. B. Van Vleck, Mr. J. H. Van Vleck, Professor Oswald Veblen, Professor W. P. Webber, Mr. F. M. Weida, Professor Mary E. Wells, Professor C. E. Wilder, Professor F. B. Wiley, Professor C. H. Yeaton, Professor J. W. A. Young.

Two courses of five lectures each were given, as follows:

I. Professor G. D. BIRKHOFF: "Dynamical systems."

II. Professor F. R. MOULTON: "Topics from the theory of functions of infinitely many variables."

Abstracts of the lectures follow below. The lectures will be published later in full as Volume VI of the Colloquium Series.

I.

LECTURE I. PHYSICAL, FORMAL, AND COMPUTATIONAL ASPECTS OF DYNAMICAL SYSTEMS.

1. The conservation of energy; rates of doing work of external forces:

$$Q_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + R_i \qquad (i = 1, 2, \cdots, n),$$

$$\sum_{i=1}^{n} \dot{q}_i R_i \equiv 0.$$

2. Lagrangian systems; $R_1 = R_2 = \cdots = R_n = 0$. Internal and external characterization.

3. Variational form of equations under no external forces: $\delta \int L dt = 0$. Change of variables.

4. Equations of variation.

5. Integrals linéar or quadratic in the velocities.

6. The Hamiltonian equations: $\dfrac{dp_i}{dt} = \dfrac{\partial H}{\partial q_i}$; $\dfrac{dq_i}{dt} = -\dfrac{\partial H}{\partial p_i}$.

Their fundamental properties. Formal series.

7. Methods of computation and their validity.

8. Relativistic dynamics.

9. Dissipative systems.

Lecture II. Types of Motions Such as Periodic and Recurrent Motions, and Motions Asymptotic to Them.

1. Existence of periodic motions; minimum and minimax methods; method of analytic continuation; other methods.
2. Hyperbolic periodic motions and their asymptotic motions.
3. Stable elliptic periodic motions.
4. Unstable elliptic periodic motions and their asymptotic motions.
5. Recurrent motions and their asymptotic motions.
6. Other types of motions.
7. The extension in General Analysis.

Lecture III. Interrelation of Types of Motion with Particular Reference to Integrability and Stability.

1. Transitivity and intransitivity.
2. Distribution of periodic motions.
3. Distribution of recurrent and other special motions.
4. Criteria for various types of integrability.
5. Non-integrability of general case.
6. Criteria for various types of stability.

Lecture IV. The Problem of Three Bodies and Its Extension.

1. The equations of motion and the classical integrals.
2. Regularization at double collision.
3. Impossibility of triple collision if area constants are not all zero.
4. Proof that for given area constants not all zero, and small initial mutual distances, the sum of the mutual distances becomes infinite.
5. Deduction of Sundman's theorem on mutual distances as corollary.
6. Extension to more general law of force.
7. Extension to case of n bodies.

LECTURE V. THE SIGNIFICANCE OF DYNAMICAL SYSTEMS FOR GENERAL SCIENTIFIC THOUGHT.

1. The dynamical model in physics.
2. Modern cosmogony and dynamics.
3. Dynamics and biological thought.
4. Dynamics and philosophical speculation.

II.

LECTURE I. INFINITE SYSTEMS OF LINEAR EQUATIONS.

1. Completely reduced systems. Historical examples.
2. The formal method of reduced systems. Historical examples.
3. Normal infinite determinants. The Hill-Poincaré form; the von Koch form.
4. Infinite systems of linear equations having normal determinants and bounded right members.
5. Absolutely convergent infinite determinants.
6. Infinite systems of linear equations having absolutely converging determinants and bounded right members.
7. Infinite systems of linear equations having absolutely converging determinants and coefficients analytic functions of a parameter.
8. Irregular solutions of infinite systems of linear equations. Examples.
9. The general theory of Schmidt. Solutions for which

$$\sum_{i=1}^{\infty} |x_i|^p \qquad (p > 1)$$

converges, including the limiting cases $p = 1$, $p = \infty$.

10. The method of successive approximations.

LECTURE II. ON PROPERTIES OF FUNCTIONS OF INFINITELY MANY VARIABLES.

1. Hilbert space (H-space) and parallelepipedon space (P-space). The mutual independence of H-space and P-space.
2. Definitions of limit points. Existence of a limit point in an infinite set of points.
3. Types of continuity and relations among them.
4. Convergent functions and uniformly convergent functions.

5. Independence of continuity and convergence.
6. Representation of convergent functions by series.
7. A continuous function of infinitely many variables in a closed P-space has a maximum and a minimum and all intermediate values.
8. Definition of analytic functions of infinitely many variables.
9. Definition of normal functions of infinitely many variables.
10. Finite operations on functions of infinitely many variables.
11. Limiting processes on functions of infinitely many variables.
12. The mean-value theorem for completely continuous functions having first derivatives.
13. Taylor's theorem for functions of infinitely many variables.

Lecture III. Infinite Systems of Implicit Function Equations.

1. Analytic solutions of reduced normal equations.
2. Analytic continuation of the solutions.
3. Solutions of normal equations having normal determinants of the coefficients of the linear terms of the dependent variables.
4. Properties of the solutions of normal equations.
5. Solution of reduced normal equations by the method of successive approximations.
6. Extension of the solution to a boundary of the region of definition of the equations.
7. Solutions of infinite systems of equations having continuous first derivatives.
8. Properties of the solutions.
9. Extension of the solution to a boundary of the region of definition of the equations.

Lecture IV. Infinite Systems of Differential Equations.

1. Analytic solutions of normal equations.
2. Solution of normal equations by the method of successive approximations.

3. Solution of equations satisfying the Lipschitz condition by the method of successive approximations.
4. Properties of the solutions.
5. Extension of the solution to a boundary of the region for which the equations are defined.
6. Solution by the Cauchy-Lipschitz method.
7. Solutions of infinite systems of linear differential equations having constant coefficients.
8. Solutions of infinite systems of linear differential equations having periodic coefficients.

LECTURE V. APPLICATIONS OF FUNCTIONS OF INFINITELY MANY VARIABLES.

1. Hill's problem of the motion of the lunar perigee.
2. Solutions of linear differential equations in the vicinity of singular points.
3. The determination of the moon's variational orbit.
4. Determination of periodic solutions of certain finite systems of differential equations.
5. The dynamics of a certain type of infinite universe.

At the close of the colloquium, Professor E. B. Van Vleck expressed the appreciation of those present for the excellence of the lectures, and tendered the thanks of the American Mathematical Society to the University of Chicago for the generous provision it had made for the colloquium, and for the welfare of the participants. An appropriate reply was made by Professor E. H. Moore.

W. A. HURWITZ.

NOTE ON VELOCITY SYSTEMS IN CURVED SPACE OF N DIMENSIONS.

BY PROFESSOR JOSEPH LIPKA.

(Read before the American Mathematical Society April 24, 1920.)

§ 1. *Introduction.*

IN a previous paper,* the author gave a complete geometric characterization of the families of curves (termed natural

* "Natural families of curves in a general curved space of n dimensions," *Trans. Amer. Math. Society*, vol. 13 (1912), pp. 77–95. We shall hereafter refer to this paper as "Natural families."

families) defined as the extremals connected with variation problems of the form

$$(1) \qquad \int F ds = \text{minimum},$$

where F is any point function and ds is the element of arc in the space considered. Such a system consists of $\infty^{2(n-1)}$ curves, one through each point in each direction. Among the dynamical systems whose determination leads to an integral of this form we may mention: (1) the trajectories in a conservative field of force for a given constant of energy h, where $F = \sqrt{W + h}$, W being the work function (negative potential); (2) the brachistochrones under conservative forces, $F = 1/\sqrt{W + h}$; (3) the forms of equilibrium of a homogeneous, flexible, inextensible string acted on by conservative forces (general catenaries), $F = W + h$; the paths of light in an isotropic medium, $F = \nu$, the variable index of refraction.

The complete characteristic geometrical properties of a natural family in any curved space, V_n, are:*

(A_1) The locus of the centers of geodesic curvature of the ∞^{n-1} curves which pass through any point of V_n is a euclidean space of $n - 1$ dimensions (S_{n-1}).

(A_2) The osculating geodesic surfaces (V_2's) at any point of V_n form a bundle of surfaces, i.e., all contain a fixed direction (and hence the geodesic in that direction) which is normal to the S_{n-1} of property A_1.

(B) The n directions at any point of V_n, in which, as a consequence of property A (i.e., A_1 and A_2), the osculating geodesic circles (circles of constant geodesic curvature) hyperosculate the curves of the given family, are mutually orthogonal.

Now, property A alone completely characterizes a much wider class of curves, designated as a velocity system.† We have pointed out several dynamical problems which lead to a system of curves characterized by properties A and B. It is the purpose of this note to point out a dynamical problem which leads to the more general system of curves characterized by property A alone.‡

* "Natural families," p. 78.

† This was designated as a system of type (G) in "Natural families," pp. 85–86.

‡ See the discussion of the problem for a euclidean space of 3 dimensions by E. Kasner, Princeton Colloquium Lectures, p. 42.

§ 2. *Differential Equations of Velocity Systems.*

Let us express our problem analytically. If the element of arc length in a general curved space V_n is given by*

$$ds^2 = \sum_{ik} a_{ik}dx_i dx_k \tag{2}$$

and we use s as the parameter along our curves, the differential equations of any natural system (characterized by properties A and B) are†

$$x_i'' + \sum_{\lambda\mu} \left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\} x_\lambda' x_\mu' = \sum_l \frac{\partial(\log F)}{\partial x_l}(A_{il} - x_i'x_l') \qquad (i = 1, 2, \cdots, n), \tag{3}$$

where A_{il} denotes the minor of a_{il} in the determinant $a = |a_{\lambda\mu}|$ divided by a itself, and we have used the Christoffel symbols

$$\left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\} = \sum_l A_{il} \left[\begin{matrix} \lambda\mu \\ l \end{matrix} \right]; \qquad \left[\begin{matrix} \lambda\mu \\ l \end{matrix} \right] = \sum_i a_{il} \left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\}; \tag{4}$$

$$\left[\begin{matrix} \lambda\mu \\ l \end{matrix} \right] = \frac{1}{2}\left(\frac{\partial a_{\lambda l}}{\partial x_\mu} + \frac{\partial a_{\mu l}}{\partial x_\lambda} - \frac{\partial a_{\lambda\mu}}{\partial x_l} \right).$$

On the other hand, the differential equations of a velocity system (characterized by property A) are‡

$$x_i'' + \sum_{\lambda\mu} \left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\} x_\lambda' x_\mu' = \sum_l \phi_l(A_{il} - x_i'x_l') \qquad (i = 1, 2, \cdots, n), \tag{5}$$

where the ϕ's are arbitrary point functions. This system will reduce to a natural system if

$$\phi_l = \frac{\partial(\log F)}{\partial x_l} \qquad (l = 1, 2, \cdots, n), \tag{6}$$

i.e., the ϕ's are the partial derivatives with respect to x of a single function. This is the analytic equivalent of property B.

* We write only Σ_{ik} and understand that the summation is to be carried out from 1 to n for each of the indicated subscripts.

† "Natural families," p. 80. Throughout this paper, primes refer to total derivatives with respect to arc length s, while dots refer to total derivatives with respect to time t.

‡ "Natural families," p. 85.

§ 3. *Dynamical Interpretation of Velocity System.*

Let us consider the motion of a particle in a curved n-space V_n under any positional forces. We start with the Lagrangian equations of motion,*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_l}\right) - \frac{\partial T}{\partial x_l} = X_l \quad (l = 1, 2, \cdots, n), \tag{7}$$

where T is the kinetic energy, given by

$$T = \tfrac{1}{2}\sum_{ik} a_{ik}\dot{x}_i\dot{x}_k, \tag{8}$$

and the X's are the components of force given as functions of the coordinates $x_1, x_2, \cdots, x_n$. Equations (7) may be expanded as

$$\begin{aligned} X_l &= \frac{d}{dt}\left(\sum_k a_{lk}\dot{x}_k\right) - \frac{1}{2}\sum_{ik}\frac{\partial a_{ik}}{\partial x_l}\dot{x}_i\dot{x}_k, \\ &= \sum_k a_{lk}\ddot{x}_k + \sum_{ik}\frac{\partial a_{lk}}{\partial x_i}\dot{x}_i\dot{x}_k - \frac{1}{2}\sum_{ik}\frac{\partial a_{ik}}{\partial x_l}\dot{x}_i\dot{x}_l, \\ &= \sum_k a_{lk}\ddot{x}_k + \sum_{ik}\begin{bmatrix} ik \\ l \end{bmatrix}\dot{x}_i\dot{x}_k. \end{aligned}$$

Multiplying by A_{ml}, summing with respect to l, and employing

$$\sum_i a_{ik}A_{il} = \begin{Bmatrix} 0, \text{ if } k \neq l \\ 1, \text{ if } k = l \end{Bmatrix}, \tag{9}$$

we get

$$\ddot{x}_m + \sum_{ik}\begin{Bmatrix} ik \\ m \end{Bmatrix}\dot{x}_i\dot{x}_k = \sum_l A_{ml}X_l \quad (m = 1, 2, \cdots, n). \tag{10}$$

These equations give us the components of acceleration along a curve as functions of the coordinates and the components of velocity.

Since the velocity along the curve is given by

$$\dot{s}^2 = \sum_{ik} a_{ik}\dot{x}_i\dot{x}_k, \tag{11}$$

we may, by differentiation, get the acceleration along the path; thus

* See E. T. Whittaker, Analytical Dynamics, p. 39.

$$
(12)\qquad
\begin{aligned}
\dot{s}\ddot{s} &= \sum_{ik} a_{ik}\dot{x}_i\ddot{x}_k + \frac{1}{2}\sum_{ikr}\frac{\partial a_{ik}}{\partial x_r}\dot{x}_i\dot{x}_k\dot{x}_r \\
&= \sum_{ik} a_{ik}\dot{x}_i\left(\sum_l A_{kl}X_l - \sum_{\alpha\beta}\begin{Bmatrix}\alpha\beta\\ k\end{Bmatrix}\dot{x}_\alpha\dot{x}_\beta\right) + \frac{1}{2}\sum_{ikr}\frac{\partial a_{ik}}{\partial x_r}\dot{x}_i\dot{x}_k\dot{x}_r \\
&= \sum_l \dot{x}_l X_l - \sum_{i\alpha\beta}\begin{bmatrix}\alpha\beta\\ i\end{bmatrix}\dot{x}_i\dot{x}_\alpha\dot{x}_\beta + \frac{1}{2}\sum_{i\alpha\beta}\frac{\partial a_{i\alpha}}{\partial x_\beta}\dot{x}_i\dot{x}_\alpha\dot{x}_\beta \\
&= \sum_l \dot{x}_l X_l,
\end{aligned}
$$

the reductions being accomplished by using (4) and (9).

As a first step in getting the differential equations of the trajectories, we have

$$x_m' = \frac{\dot{x}_m}{\dot{s}}; \qquad x_m'' = \frac{\dot{s}\ddot{x}_m - \dot{x}_m\ddot{s}}{\dot{s}^3};$$

and using (10) and (12), we get

$$x_m'' = \frac{1}{\dot{s}^2}\sum_l A_{ml}X_l - \sum_{ik}\begin{Bmatrix} ik\\ m\end{Bmatrix} x_i'x_k' - \frac{1}{\dot{s}^2}x_m'\sum_l x_l'X_l,$$

or

$$(13)\qquad x_m'' + \sum_{ik}\begin{Bmatrix} ik\\ m\end{Bmatrix} x_i'x_k' = \frac{1}{\dot{s}^2}\sum_l X_l(A_{ml} - x_m'x_l') \qquad (m = 1, 2, \cdots, n).$$

To get the differential equations of the trajectories we should have to eliminate the speed $\dot{s}$; this would lead to a set of equations of the third order representing ∞^{2n-1} curves. But, for our purpose, we need not go any further. Equations (13) hold for any trajectory, and along this the speed $\dot{s}$ varies from point to point. Now if in (13) we replace $1/\dot{s}^2$ by a constant c, we get

$$(14)\qquad x_m'' + \sum_{ik}\begin{Bmatrix} ik\\ m\end{Bmatrix} x_i'x_k' = c\sum_l X_l(A_{ml} - x_m'x_l') \qquad (m = 1, 2, \cdots, n),$$

a set of differential equations of the second order representing a system of $\infty^{2(n-1)}$ curves (called a velocity system) one through each point in each direction. This system may therefore be defined dynamically as follows:

A curve is a velocity curve corresponding to the speed $\dot{s}_0$ *if a particle starting from a point of such a curve and in the direction of the curve and with that speed describes a trajectory osculating the curve.*

Now, we note that equations (14) are, with a change in subscripts, exactly equations (5), where

$$\phi_l = cX_l = \frac{1}{\dot{s}^2}X_l \qquad (l = 1, 2, \cdots, n). \tag{15}$$

We have thus formulated a dynamical problem which leads to the system of curves (called a velocity system) characterized geometrically by property A. For each constant value assigned to the speed $\dot{s}$, we get a velocity system, and the totality of ∞^1 systems obtained by varying $\dot{s}$ constitute a complete velocity system of ∞^{2n-1} curves in V_n.

§ 4. *Velocity Systems and Natural Systems.*

Velocity systems are not in general systems of trajectories, brachistochrones, or catenaries, but if the field of force is conservative, then

$$X_l = \frac{\partial W_1}{\partial x_l} \qquad (l = 1, 2, \cdots, n), \tag{16}$$

where W_1 is the work function defining the field, and, as pointed out in § 2, the velocity system corresponding to a speed $\dot{s}_0$ becomes a natural system defined by equation (1) or by the point function F, where by (6)

$$\phi_l = \frac{\partial(\log F)}{\partial x_l},$$

and by (15)

$$c\frac{\partial W_1}{\partial x_l} = \frac{\partial(\log F)}{\partial x_l};$$

hence

$$F = e^{W_1/\dot{s}^2} \tag{17}$$

On the other hand, in a conservative field of force with work function W_2 and given constant of energy h, the natural system defined by equation (1) or by the point function F is a system of

$$
(18) \qquad \begin{array}{lll} \text{trajectories,} & \text{if } F = \sqrt{W_2 + h}, \\ \text{brachistochrones,} & \text{if } F = 1/\sqrt{W_2 + h}, \\ \text{catenaries,} & \text{if } F = W_2 + h. \end{array}
$$

By comparison of (17) and (18), we may now state:

A velocity system for the speed $\dot{s}_0$ *in a conservative field with work function* W_1 *is a system of* (1) *trajectories,* (2) *brachistochrones,* (3) *catenaries for the constant of energy* h *in a conservative field with work function* W_2, *where*

$$
(1)\ W_2 = e^{2W_1/\dot{s}_0^2} - h, \qquad (2)\ W_2 = e^{-(2W_1/\dot{s}_0^2)} - h,
$$

$$
(3)\ W_2 = e^{W_1/\dot{s}_0^2} - h.
$$

Since $W_1 =$ constant gives $W_2 =$ constant, the two fields have the same equipotential hypersurfaces and the same lines of force.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
February, 1920.

AUGUSTUS DE MORGAN ON DIVERGENT SERIES.

BY PROFESSOR FLORIAN CAJORI.

(Read before the San Francisco Section of the American Mathematical Society April 10, 1920.)

SEVERAL English mathematicians writing in the second quarter of the nineteenth century disapproved of the banishment of divergent series which had been brought about by the followers of A. L. Cauchy and N. H. Abel. These protests were unheeded, doubtless because they were not accompanied by indications disclosing how divergent series could be used with safety. There was one exception, however: Augustus De Morgan reached results which, had they been followed up promptly, might have re-introduced divergent series thirty years earlier than was actually the case. De Morgan's researches have been overlooked in historical statements, except by H. Burkhardt,* who, however, missed the parts of De Morgan which foreshadow a new theory.

* H. Burkhardt "Ueber den Gebrauch divergenter Reihen in der Zeit von 1750–1860," *Math. Annalen*, vol. 70 (1911), pp. 169–206. This article contains much minute information regarding many writers.

Not only did De Morgan issue a vigorous protest against the abandonment of divergent series, but he laid the foundation to three parts of the subject as it stands to-day: (1) A principle of divergent series recently enunciated more clearly by G. H. Hardy; (2) a summation formula more general than that of Cauchy; (3) the use of asymptotic series.

Omitting many clever observations made by De Morgan, we confine ourselves to these three parts. There is considerable incompleteness in De Morgan's development of them, especially of the first part.

1. *A Principle of Divergent Series.*—De Morgan's point of view is partly contained in the following assumption*: "If then V be expanded into the series $P_0 + P_1 + P_2 + \cdots$ and if the sum of n terms, $P_0 + P_1 + \cdots + P_{n-1}$ be called Q_n; we obviously have

$$\int_0^a V\,dv = \int_0^a P_0 dv + \int_0^a P_1 dv + \cdots + \int_0^a (V - Q_n)dv,$$

where n is made infinite after integration. When the series $P_0 + P_1 + \cdots$ is convergent, then, even granting that $\int(V - Q_\infty)dv$ may have circumstances peculiar to $n = \infty$, it is of no consequence, since considerations of form are rendered useless by evanescence of value: the elements of $\int(V - Q_n)dv$ must, by the hypothesis of convergency, diminish without limit as compared with the corresponding elements of $\int P_0 dv$, $\int P_1 dv$, etc. Even if integration converted the convergent series into a divergent one, this would still be the case."

In this rather difficult passage, De Morgan considers

$$\int_0^a \lim_{n\to\infty} (V - Q_n)dv \text{ and } \lim_{n\to\infty} \int_0^a (V - Q_n)dv.$$

If $P_0+P_1+\cdots$ is convergent, then, even if this infinite series becomes divergent on integration, he finds $\int_0^a \lim_{n\to\infty} (V-Q_n)dv=0$,

* A. De Morgan "On divergent series, and various points in analysis connected with them" in *Cambridge Philosophical Society, Transactions*, vol. 8, Part II, pp. 182–203; see page 189. The paper bears the date of Jan. 15, 1843; it was read March 4, 1844. De Morgan treated divergent series also in vol. 11, pp. 190–202, of the above *Transactions* and in his Differential and Integral Calculus, London, 1842, chapters 19 and 20, but he did not reach noteworthy results other than those given in his article of 1844.

and the elements of $\int_0^a (V - Q_n)dv$ must "diminish without limit" as "n is made infinite after integration." It would seem that $\int_0^a \lim_{n\to\infty} (V - Q_n)dv$ is used here for the purpose of interpreting $\lim_{n\to\infty} \int_0^a (V - Q_n)dv$. No example is given of the conversion of a convergent into a divergent series by integration, but he takes

$$(1)\quad \frac{1 - c\cos av}{1 - 2c\cos av + c^2} = 1 + c\cos av + \cdots + c^n \cos nav + R_{n+1},$$

then "neglects" R_{n+1} and lets $n = \infty$. Thus he obtains an infinite series which is divergent for $c > 1$. Multiplying both sides by $e^{-v^2}dv$ and integrating from $v = 0$ to $v = \infty$, he gets a convergent series. He shows this result to be arithmetically wrong.

Reversing the order of procedure and retaining R_{n+1} in (1), he integrates first and then lets $n = \infty$; he obtains a convergent result which is arithmetically correct. We have here a distant approach to the principle advanced by G. H. Hardy* nearly sixty years later, to the effect that, in such cases, "we may use the otherwise meaningless expression" first obtained "as a formal equivalent for the determinate expression" obtained last.

2. *A Summation Formula More General than that of Cauchy.*—That De Morgan in 1843 held views which were in advance of his time is evident from the following quotation:† "In every convergent series, the limit of the sum of all its terms *is* the mean value obtained from all the summations: the mean of n partial summations $A_1, (A_1 + A_2), \cdots, (A_1 + A_2 + \cdots + A_n)$ is

$$A_1 + \frac{n-1}{n} A_2 + \frac{n-2}{n} A_3 + \cdots + \frac{1}{n} A_n,$$

which, as n is increased without limit, has $A_1 + A_2 + \cdots$ *ad inf.* for its limit. Hence, by Poisson's principle, by which

* *Cambridge Philosophical Society, Transactions*, vol. 19, 1900–1904, p. 297. See also T. J. I. Bromwich, Infinite Series, London, 1908, p. 267, § 99.

† *Cambridge Philosophical Society, Transactions*, vol. 8, Part II, 1844, p. 192.

I mean the assumption of the right to apply the maxim, 'that which is quantitatively true up to the limit, is true in the same sense at the limit, when the limit presents an *incalculable form*'—we may assert most positively, that $1 - 1 + 1 - \cdots$ must be $\frac{1}{2}$ whenever it is the limiting form of convergency: not on the metaphysical doctrine (probably suggested by the known result) of Leibnitz, namely, that we can see no reason to prefer 0 to 1, or 1 to 0, and must therefore take a mean; but because n partial summations give the mean $1/n \times n/2$ or $1/n \times (n + 1)/2$ according as n is even or odd, and the limit of both is $\frac{1}{2}$."

De Morgan's definition of "sum" is substantially the same as a definition given by G. Frobenius* in 1880 and the first of several definitions given by E. Cesàro† in 1890. De Morgan pointed out that, when applied to convergent series, his formula yields the same results as does the restricted formula ordinarily used for convergent series; that is, he recognized the need of "consistency" of definition.

3. *The Use of Asymptotic Series.*—As De Morgan's paper is dated January 15, 1843, he had not seen Cauchy's article‡ printed later in the same year in which Stirling's divergent series for $\log \Gamma(x)$ is used for computing $\log \Gamma(x)$ for large positive values of x. De Morgan appreciates the importance of asymptotic series, but does not compute the asymptotic value of any one series as was done by Cauchy. Instead, De Morgan is groping after a general theory. He says:§ "When an alternating series is convergent, and a certain number of its terms are taken as an approximation, the first term neglected is a superior limit of the error of approximation. This very useful property was *observed* to belong to large classes of alternating series, when finitely or even infinitely divergent: I do not remember that any one has *denied* that it is universally true, while many have implicitly asserted it. When the series is convergent for a certain number of terms, particularly if the terms become very small before they begin to increase again, it obviously makes the divergent alternating

* G. Frobenius, "Ueber die Leibnitzsche Reihe," *Jour. für Math.*, vol. 89 (1880), p. 262.

† E. Cesàro, "Sur la multiplication des séries," *Bull. des Sciences Math.* (2), vol. 14 (1890), p. 119.

‡ Cauchy, *Comptes Rendus*, vol. 17 (28 août, 1843), p. 370; Œuvres, Série I, vol. 8, p. 18.

§ *Cambridge Philosophical Society, Transactions*, vol. 8, Part II, p. 193.

series practically as useful as the converging series, perhaps even more so, for it is very frequent that the greater the ultimate divergence, the greater also is the primitive tendency towards convergence."

The theorem that "the first term neglected is a superior limit of the error of approximation," though, as De Morgan says, not universally true, is true, he says, of large classes of alternating series, including the series $\phi(x) - \phi(x + 1) + \phi(x + 2) - \cdots$ "for all cases in which $\phi(x)$ can be the expressed by $\int_\alpha^\beta e^{nvx} X_v dv$, X_v being always positive between limits."

In the development of the modern theories of divergent series, Augustus De Morgan deserves to be ranked as a pioneer.

On December 23, 1857, Sir William R. Hamilton* wrote to De Morgan: "About diverging series, you know a great deal more than I do. In fact you are aware that I early conceived a sort of prejudice against them, in consequence of some of Poisson's remarks. Counter-remarks of yours had staggered me, but had not been carefully weighed. At last (and, I regret to say it, without having yet found the Papers by you and Stokes on *such* series, for Stokes, or Adams for him, sent me about a month ago a duplicate of his memoir on the numerical calculation of the values of certain definite integrals, having a great affinity to my last Paper) I am become a convert to those Divergents; so far at least as to be satisfied that in an extensive class of cases, and with suitable limitations, they may be safely and advantageously used."

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RUSSELL'S INTRODUCTION TO MATHEMATICAL PHILOSOPHY.

Introduction to Mathematical Philosophy. By BERTRAND RUSSELL. (The Library of Philosophy.) London, Allen and Unwin, and New York, The Macmillan Company, 1919. 8vo. viii + 208 pp. $3.00.

THIS book, called an introduction to mathematical philosophy, is an excellent introduction to that field and, more

* R. P. Graves, Life of Sir William Rowan Hamilton, vol. 3, 1899, p. 538.

particularly, to mathematical logic. In the preface the author brings out the fact that mathematical logic is relevant to philosophy and "for this reason, as well as on account of the intrinsic importance of the subject, some purpose may be served by a succinct account of the main results of mathematical logic in a form requiring neither a knowledge of mathematics nor an aptitude for mathematical symbolism."

The first chapter is concerned with the logical basis of the series of natural numbers. The system of postulates of Peano is discussed in some detail. The postulates used in "arithmetization" are indefinite and there is an increase in definiteness produced by "logicizing" mathematics. We cannot, by Peano's method, explain what we mean by the undefined terms 0, number, and successor in terms of simpler concepts although we may know what we mean by them. Russell says: "It is quite legitimate to say this (the last statement) when we must, and at *some* point we all must; but it is the object of mathematical philosophy to put off saying it as long as possible. By the logical theory of arithmetic we are able to put it off for a very long time."

In Chapters II and III the logical theory of the natural numbers is developed. Chapter II contains an exposition of the definition of cardinal number given by Frege, i.e., the cardinal number of a class is the class of all those classes that are similar to it. The following chapter is headed "Finitude and Mathematical Induction." There the definitions of 0, successor, hereditary property, hereditary class, inductive property, inductive class and the posterity of a natural number are given in terms of elemental logical concepts. The "natural numbers" are defined as the posterity of 0 with respect to the relation "immediate predecessor." The idea back of this procedure is that of mathematical induction. Russell emphasizes the fact that mathematical induction is a definition and not a principle. "There are some numbers to which it can be applied and there are others to which it cannot be applied. We *define* the 'natural numbers' as those to which proofs by mathematical induction can be applied, i.e., as those that possess all inductive properties." For this reason the author prefers the term "inductive numbers" to "natural numbers." Of course, this point of view is legitimate on the basis of the procedure above outlined. With respect to another setting up of the natural numbers mathematical induction might well be a principle.

Chapter IV is devoted to order relations and Chapters V and VI to relations in general. In the treatment of order the three kinds of relations, asymmetrical, transitive and connected, are defined preliminary to giving the following definition: A series or a serial relation is a relation which is asymmetrical, transitive and connected. On the basis of these definitions it is shown how the "natural numbers" can be ordered serially. Other examples of series are also given. Relations which do not have the three characteristic properties of serial relations are discussed and the chapter closes with a brief account of series for which the defining relation is between more than two terms. The relation "between" is discussed in some detail. Chapter V treats in a general way of relations. Neither here nor anywhere else in the book is "relation" defined. A clear cut definition of relation, say as a correspondence or as the underlying propositional function, and its discussion would seem to be essential to a treatment of relations such as given in this book. The author does not make this omission in the Principia (volume 1) and there seems to be no good reason for making it here. The necessary material is right at hand. The material of the present chapter is largely a repetition of matter which appeared in previous chapters, but its importance warrants repetition. Relations of the following kinds are considered: asymmetrical, transitive, connected, ancestral, one-one, one-many, many-one, many-many. In discussing the similarity of relations in Chapter VI the following two definitions are fundamental: "A relation S is said to be a correlator of two relations P and Q if S is one-one, has the field of Q for its converse domain and is such that P is the relative product of S and Q and the converse of S." "Two relations P and Q are similar if there exists at least one correlator of P and Q." When two relations are similar they share all properties which do not depend upon the actual terms in their fields. In this connection the question arises: "Given some statement in a language of which we know the grammar and the syntax, but not the vocabulary, what are the possible meanings of such a statement, and what are the meanings of the unknown words that would make it true?" This question is important because "it represents, much more nearly than might be supposed, the state of our knowledge of nature."

The notion of similarity leads to the concept "the relation

number of a given relation": the class of all those "relations that are similar to the given relation." The ordinal numbers are special cases of relational numbers. The chapter closes with an interesting application to the philosophical speculation concerning a comparison between an objective and a subjective world.

In Chapter VII the idea of number is extended by supplying logical definitions of rational, real and complex numbers. The author remarks that the discovery of correct definitions in this field was delayed by the common idea that each extension of number included the previous sorts as special cases. The definition of positive and negative integers, which is here given, is: "If m is any inductive number (natural number) then $+ m$ is the relation of $n + m$ to n for any n (a cardinal number) and $- m$ is the converse relation, i.e., the relation of n to $n + m$." According to this definition "$+ m$ is every bit as distinct from m as $- m$ is." A definition of a similar sort is given for positive and negative ratios. Definitions of the following terms are then given: "upper limit (lower limit) of a class α with respect to a relation P"; "maximum (minimum) of a class α with respect to a relation P"; "upper (lower) boundary of a set α." A "real number" is a segment of the series of ratios in order of magnitude. An "irrational number" is a segment of the series of ratios which has no boundary. A "rational number" is a segment of the series of ratios which has a boundary. In these definitions a segment is that class of the two determined by a Dedekind cut which contains the smaller numbers. A complex number is defined as an ordered pair of real numbers. The various arithmetical operations are defined and discussed for each particular class of numbers. The extensions in this chapter do not involve infinity.

In the next two chapters the notion of number is applied to infinite collections. On the basis of the assumption that no two inductive numbers have the same successor (given by Peano) it is shown that the number of inductive numbers is a new number not possessing all inductive properties. To quote the author again: "The difficulties that so long delayed the theory of infinite numbers were due largely to the fact that some, at least, of the inductive properties were wrongly judged to be such as *must* belong to all numbers; indeed it was thought that they could not be denied without contra-

diction. The first step in understanding infinite numbers consists in realizing the mistakenness of this view." The course of the discussion of this chapter leads naturally to the definitions: A reflexive class is one which is similar to a proper part of itself; a reflexive cardinal number is the cardinal number of a reflexive class. In order to give a definition of the number of inductive numbers the following definition of a progression is given: A progression is a one-one relation such that there is just one term belonging to the domain but not to the converse domain and the domain is identical with the posterity of this one term. The number of inductive numbers, $\aleph_0$, is the set of all domains of progressions. Some properties of $\aleph_0$ are developed. A finite class or cardinal is defined as one which is inductive and an infinite class or cardinal is one which is not inductive. The statement is made without proof that all reflexive classes are infinite (non-inductive). The reader is referred to a later chapter for the connection between the theorem that all infinite classes are reflexive and the multiplicative axiom. The higher transfinite cardinals and ordinals are briefly discussed and Chapter IX closes with a review of the formal laws obeyed by the transfinite cardinals and ordinals. A general definition of a transfinite ordinal is not given. "The importance of ordinals, though by no means small, is distinctly less than that of cardinals, and is very largely merged in that of the more general conception of relation-numbers."

Limits and continuity are the topics discussed in the next two chapters. The ordinal character of the notion of limit is emphasized. In the first of these chapters the notion of the limit of a set of elements and such related notions as minima, maxima, sequents, precedents, upper limits, lower limits and boundaries of a class with respect to a given relation are defined. A brief treatment of the Dedekind and Cantor definitions of continuous series is given at the end of the chapter. The other chapter is devoted to limits and continuity of functions and is more technical. The ordinary definitions of $\lim_{x \to a} f(x)$ and continuous function are given, though sometimes in a more abstract form.

The rest of the book is devoted to the logic (proper) of mathematics, the various topics treated becoming more and more elemental as the end of the book is reached. In Chapter

XII a very clear discussion of the multiplicative axiom is given. It is shown how this axiom or a weaker form is needed to prove such theorems as these: that any class can be well-ordered; that the sum of $\aleph_0$ classes of $\aleph_0$ members each has $\aleph_0$ members; that a non-inductive class is reflexive. The author's reaction to questions in this chapter is contained in the closing paragraph: "It is not improbable that there is much to be discovered in regard to the topics discussed in the present chapter. Cases may be found where propositions which seem to involve the multiplicative axiom can be proved without it. It is conceivable that the multiplicative axiom in its general form may be shown to be false. From this point of view, Zermelo's theorem offers the best hope: the continuum or some still more dense series *might* be proved to be incapable of having its terms well ordered, which would prove the multiplicative axiom false, in virtue of Zermelo's theorem. But so far, no method of obtaining such results has been discovered, and the subject remains wrapped in obscurity."

The subject of the next chapter is "The Axiom of Infinity and Logical Types." One form of the axiom of infinity is "If n be any inductive cardinal number, there is at least one class of individuals having n terms." Without the axiom of infinity or its equivalent the theory of real numbers and the theory of transfinite numbers would not exist. Russell spends some time showing that the axiom of infinity cannot be proved after postulating a class of individuals by forming the complete set of individuals, classes, classes of classes, etc. This kind of reasoning leads to such contradictions as the existence of the greatest cardinal number, the class of all classes, etc. The fallacy of the reasoning consists in the formation of a "class which is not pure as to type." At this point a little is said about the theory of types but the mention is too brief to be satisfying. A good brief exposition of the theory of types is probably impossible at this time. Some pertinent remarks are: "Classes are logical fictions and a statement which appears to be about a class will only be significant if it is capable of translation into a form in which no mention is made of the class." "If there are n individuals in the world and 2^n classes of individuals we cannot form a new class, consisting of both individuals and classes and having $n + 2^n$ members." The author does not pretend to have

explained the doctrine of types, but his object is to indicate why there is need for such a doctrine. Other "proofs," more or less metaphysical, of the axiom of infinity are briefly examined.

The next four chapters are the most fundamental of the book. Their object is a critique of the notion of *class*. The topics of the first three of these chapters, viz: (1) the theory of deductions and incompatibility, (2) propositional functions, (3) descriptions, although very important in themselves, are introductory to the study of the theory of classes given in the last of these chapters.

In the chapter on the theory of deduction Russell restates his thesis that "what can be known, in mathematics and by mathematical methods, is what can be deduced from pure logic." The essential part of the chapter consists of the definitions of the "truth-functions": "not-p" (negation); "p or q" (disjunction); "p and q" (conjunction); "p and q are not both true" (incompatibility); "not-p or q" (implication). All five truth-functions are not independent. Two, negation and disjunction, were chosen in the Principia Mathematica as fundamental and the others defined in terms of these. Sheffer has shown that one primitive idea is sufficient for that purpose. It is here shown that the single primitive idea of incompatibility is sufficient. An analysis of deduction is made on the basis of the five formal principles of deduction given in the Principia. A formal principle of deduction (e.g., "p or p implies p") has a double use: to serve as the premise of an inference or as a rule of deduction. A proof that the five formal principles can be reduced to one is given in detail. This single formal principle which is much more complicated, at least in statement, than any of the five is due to M. Nicod. This formal principle and two non-formal principles furnish the apparatus from which the whole theory of deduction follows "except in so far as we are concerned with deduction from or to the existence or the universal truth of propositional functions," which are studied in the next chapter. The chapter closes with an argument in support of the author's views on implication as against those of C. I. Lewis.

Of a propositional function it might be said that it is true in all cases or that it is true in some cases. The importance of this use of propositional functions is clearly pointed out

in Chapter XV. All the primitive propositions of logic as well as the principles of deduction consist of statements that certain propositional functions are always true. It is explained how the truth-functions as applied to propositions containing apparent variables can be defined in terms of the definitions and primitive ideas for propositions containing no apparent variables. For this it is found necessary to take as primitive ideas two of the following: "always," "sometimes," "not-ϕx sometimes" (or "always" as the case may be). The simpler forms of traditional formal logic really involve the assertion of all values or some values of a compound propositional function. For example: "all S is P" means "ϕx implies ψx is always true" where ϕx and ψx denote propositional functions. Russell's treatment of traditional logic leads to such results as the following: "if there are no S's then 'all S is P' and 'no S is P' will both be true, whatever P may be." Some reasons for preferring his treatment are given in convincing form.

The fundamental meaning of "existence" is contained in the following statement: If the propositional function ϕx is sometimes true we say that arguments satisfying ϕx *exist*. We may say men exist (here ϕx is x is a man) but it is nonsense to say John exists. Another instance of the use of propositional functions which we are considering is in the notions of "modality" (*necessary, possible* and *impossible*). An undetermined value of a propositional function ϕx is necessary if the function is always true, possible if sometimes true and impossible if it is never true. At the end of the chapter we have this sentence: "For clear thinking, in many diverse directions, the habit of keeping propositional functions sharply separated from propositions is of the utmost importance, and failure to do so in the past has been a disgrace to philosophy."

The theory of descriptions, treated in the next chapter, is very important from the point of view of logic and the theory of knowledge. Only those parts of the theory which are relevant to mathematics are here discussed. A proposition involving an indefinite description about "a so-and-so" is of the form "an object having the property ϕ has the property ψ" which means "The joint assertion of ϕx and ψx is not always false." It is an important point that such propositions contain no constituent represented by the phrase "a so-

and-so." Thus such propositions can be significant when there is no such thing as "a so-and-so." This is the solution of the philosophical question of "unreality" which Russell gives. The definition of propositions involving definite descriptions is: "the term satisfying ϕx satisfies ψx" which means that there is a term c such that (1) ϕx is always equivalent to 'x is c,' (2) ψc is true." The extra condition of uniqueness is added in this case.

The theory of classes, which is taken up in Chapter XVII, is concerned with the word *the* in the plural while that of definite descriptions deals with the singular meaning of that word. Because of the paradoxes involving the notion of class the latter cannot be taken as a primitive idea. It is desired to find "a definition which will assign a meaning to propositions in whose verbal or symbolic expression words or symbols apparently representing classes occur but which will assign a meaning that altogether eliminates all mention of classes from a right analysis of such propositions." The theory here outlined reduces propositions nominally about classes to propositions about the propositional functions which define them. The theory is incomplete because it is thrown back, in part, upon the incomplete theory of types. Because of this incompleteness it is found necessary to assume the axiom of reducibility: there is a type τ such that if ϕ is a function which can take a given object a as argument, then there is a function ψ of the type τ which is formally equivalent to ϕ. The fundamental definition of the theory of classes is: if ϕ is a function which can take a given object a as argument, and τ the type mentioned in the above axiom, then to say that the class determined by ϕ has the property f is to say that there is a function of type τ, formally equivalent to ϕ, and having the property f.

The final chapter "Mathematics and Logic" opens with an assertion of the Russellian thesis that logic and mathematics are identical. The proof in all detail is not given, but one is referred to the Principia. The remainder of the chapter is devoted to a discussion of what is characteristic of mathematical (or logical) propositions. Logical propositions affirm that some propositional function is always true. Specific propositions whose truth depends upon something else than the form of the propositions do not belong to mathematics but to its applications. Mathematical propositions have the characteristic described, perhaps, by the word "tautology."

This together with the fact that they can be expressed wholly in terms of variables and logical constants (invariants under all changes of the constituents of a proposition) gives a definition of logic or mathematics. The indefiniteness of the notion of tautology leaves this definition unsatisfactory and demands improvement. At this point, however, we reach the frontier of knowledge. It is pointed out that because of the insufficiency of language a symbolism for logic is necessary. At the close of the chapter the author expresses the hope that some who have read this book will master the symbolism of symbolic logic and then help to push back still further the frontier of knowledge, especially by a new treatment of the traditional problems of philosophy.

Throughout the exposition is clear and the style fluent. Here and there a refreshing bit of humor or sarcasm is thrown in. The book is written in a way which is as simple and untechnical as it is, perhaps, possible to be without sacrificing accuracy. Besides ably serving the purpose for which it was written, the book should be very useful to anyone who has a fundamental interest in science.

The following obvious errata in the text have been noted:

page 21, line 3 from bottom, "less than 1000" should read "not less than 1000";
page 33, line 7 from top, "but is asymmetrical" should read "but is not asymmetrical";
page 160, line 19 from top, "x is a mortal man" should read "x is not a mortal man";
page 161, line 2 from top, "to which x belongs" should read "to which ϕx belongs";
page 165, line 7 from top, "(where a is a term satisfying x)" should read "(where a is a term satisfying ϕx)";
page 171, line 5 from top, "the propositional function x" should read "the propositional function ϕx";
page 176, line 3 from bottom, "a propositional function x" should read "a propositional function ϕx";
page 176, line 1 from bottom, "the value of x" should read "the value of ϕx".

G. A. PFEIFFER.

NOTES.

The twenty-seventh annual meeting of the American Mathematical Society will be held in New York City on Tuesday and Wednesday, December 28–29. At this meeting President Morley will deliver his retiring address, the subject of which will be "Pleasant Questions and Wonderful Effects." The annual election of officers and other members of the Council will close on Wednesday morning. The regular western meeting of the Society, being the forty-sixth regular meeting of the Chicago Section, will be held at the University of Chicago, on December 29–30, in conjunction with the meetings of the Mathematical Association of America and the American Association for the Advancement of Science.

The fifth summer meeting of the Mathematical Association of America was held at the University of Chicago on September 6, immediately preceding the meeting and colloquium of the American Mathematical Society. The attendance included 113 members. The following papers were read: "On certain fundamental principles in the mathematics of life insurance," by D. F. Campbell; "Certain features of the application of Makeham's laws of mortality," by H. L. Rietz; "The plan of pensions and insurance recommended by the Carnegie Foundation for the advancement of teaching," by E. L. Dodd; Report of progress of the National committee on mathematical requirements, by J. W. Young; "The debt of mathematics to the experimental sciences," by A. C. Lunn, followed by discussion, led by E. H. Moore and M. W. Haskell; "Retrospect and prospect for mathematics in America" (retiring presidential address), by H. E. Slaught. Steps were taken towards the incorporation of the Association, which was completed under the laws of the state of Illinois in September. The Association held a joint dinner with the American Mathematical Society on Tuesday evening at the Quadrangle Club.

The July number (volume 21, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "On the representation of a number as the sum of any number of squares, and in particular of five," by G. H. Hardy;

"A memoir upon formal invariancy with regard to binary modular transformations. Invariants of relativity," by O. E. Glenn; "Properties of the subgroups of an abelian prime power group which are conjugate under its group of isomorphisms," by G. A. Miller; "On the order of magnitude of the coefficients in trigonometric interpolation," by Dunham Jackson; "Concerning simple continuous curves," by R. L. Moore; "On the iteration of rational functions," by J. F. Ritt.

The following fifteen doctorates with mathematics as major subject were conferred by American universities in the academic year 1919–1920; the title of the dissertation is added in each case: J. D. Bond, Michigan, "Plane trigonometry in Richard Wallingford's Quadri partium de sinibus demonstratis;" J. Douglas, Columbia, "On certain two-point properties of general families of curves"; T. C. Fry, Wisconsin, "The use of divergent integrals in the solution of differential equations"; Gladys Gibbens, Chicago, "Comparison of different line-geometric representations for functions of a complex variable"; C. F. Green, Illinois, "On the summability and regions of summability of a general class of series of the form $\Sigma c_n g(x + n)$"; J. W. Lasley, Chicago, "Some special cases of the flecnode transformation of ruled surfaces"; Elsie J. McFarland, California, "On a special quartic curve"; J. J. Nassau, Syracuse, "Some theorems in alternates"; C. A. Nelson, Chicago, "Conjugate systems with conjugate axis curves"; E. L. Post, Columbia, "Introduction to a general theory of elementary propositions"; Susan M. Rambo, Michigan, "The point at infinity as a regular point of certain difference equations of the second order"; L. L. Steimley, Illinois, "On a general class of series of the form $Y(x) = C_0 + \Sigma C_n g(nx)$"; J. L. Walsh, Harvard, "On the location of the roots of the jacobian of two binary forms"; R. Woods, Illinois, "The elliptic modular functions associated with the elliptic norm curve E^7"; T. Yang, Syracuse, "A problem in differential geometry."

The Library of the American Mathematical Society is in receipt of the thirteenth and final volume of the magnificent edition of the complete works of Christian Huygens issued by Nijhoff on behalf of the Dutch Society of Sciences. The

first ten volumes, containing the correspondence of Huygens during the period 1636–1695, were published from 1888 to 1905. The eleventh volume, appearing in 1908, was a comparatively small one of less than four hundred pages, and contained early mathematical works of Huygens of the period 1645–1651. The twelfth volume (1910), also a small one, was devoted to works on pure mathematics, 1652–1656. The thirteenth volume (1916), of over nine hundred pages, contains the various writings of Huygens on dioptrics during the period 1653–1692. It is doubtful if the works of any other scientist have received such detailed editing in the way of translations, footnotes, and all sorts of indexes.

In the department of mathematics of the University of Michigan the following promotions are announced: Professors W. H. Butts and T. R. Running to full professorships; Professors C. E. Love and T. H. Hildebrandt to associate professorships; Dr. L. J. Rouse, Dr. A. L. Nelson, Dr. W. W. Denton, and Dr. R. B. Robbins to assistant professorships. Dr. Robbins has been granted leave of absence for the year. Mr. N. Anning, Mr. S. E. Field, Mr. K. W. Halbert, Mr. G. D. Jones, Mr. J. N. Landis, and Mr. H. A. Simmons have been appointed instructors.

Professor J. M. Taylor, of Colgate University, has retired after fifty years of teaching.

Professor H. E. Buchanan, of the University of Tennessee, has been appointed head of the department of mathematics at Tulane University.

Professor E. W. Brown, of Yale University, has been granted leave of absence for the first half year. His address will be Christ's College, Cambridge.

Dr. A. S. Hathaway, professor of mathematics at the Rose Polytechnic Institute since 1891, has retired from active teaching. He is succeeded by Dr. I. P. Sousley, of Pennsylvania State College.

At the University of Iowa, assistant professor E. W. Chittenden has been promoted to an associate professorship

of mathematics; Dr. W. H. WILSON has been promoted to the rank of associate; Dr. ROSCOE WOODS, of the University of Illinois, has been appointed instructor in mathematics, and Mr. H. M. JEFFERS, of the Lick Observatory, instructor in mathematics and astronomy.

AT the University of Wisconsin, Professor C. S. SLICHTER has been appointed dean of the graduate school; associate professor E. B. SKINNER has been promoted to a full professorship; assistant professor H. W. MARCH has been promoted to an associate professorship; Dr. WARREN WEAVER has been appointed assistant professor; Mr. HAROLD DAVIS, of Harvard University, and Mr. M. L. MACQUEEN, of Southwestern Presbyterian University, have been appointed instructors in mathematics.

ASSISTANT professor EUGENE TAYLOR, of the University of Wisconsin, has been appointed professor and head of the department of mathematics at the University of Idaho.

PROFESSOR D. A. ROTHROCK has been elected dean of the college of liberal arts of Indiana University.

AT the University of Kentucky, Dr. F. ELIZABETH LE-STOURGEON, of Carleton College, has been appointed assistant professor of mathematics, and Mr. W. E. PAYNE has been appointed instructor. Dr. G. W. SMITH has resigned, to accept an assistant professorship of mathematics at the University of Kansas.

PROFESSOR PAUL BACHMANN died at Weimar, March 31, 1920, at the age of eighty-three years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BONOLA (R.). Die nichteuklidische Geometrie. Historisch-kritische Darstellung ihrer Entwicklung. Autorisierte deutsche Ausgabe, besorgt von H. Liebmann. 2te Auflage. Leipzig, Teubner, 1919. M. 6.40

DRAEGER (M.). Ueber rekurrente Reihen von höherer, insbesondere von der dritten Ordnung. (Dissertation.) Jena, 1919.

FISCHER (P.). Koordinatensysteme. 2te, verbesserte Auflage. Berlin, 1919. M. 1.25

FRICKE (R.). Hauptsätze der Differential- und Integralrechnung. 7te Auflage. Braunschweig, 1919. M. 7.60

INGHIRAMI (G.). Table des nombres premiers et de la décomposition des nombres de 1 à 100,000, revue et corrigée par le P. V. Prompt et suivie de la table des bases des nombres tessaréens de 1 à 20,000. Paris, Gauthier-Villars, 1919. 8vo. 20 + 36 pp. Fr. 7.50

LIEBMANN (H.). See BONOLA (R.).

MANN (W. D.). Das Preisproblem der Potenzenreihe $x^\lambda + y^\lambda - z^\lambda$ in Beziehung zu verwandten Problemen und Tatsachen der Gegenwart. Mexico, Deutsche Buchhandlung, 1919. $1.25

PAULY (J.). Notions élémentaires du calcul différentiel et du calcul intégral. Paris et Liège, Béranger, 1920. 8vo. 330 pp. Fr. 20.00

PRÖLSS (O.). Graphisches Rechnen. Leipzig, Teubner, 1920. M. 1.60

PROMPT (P. V.). See INGHIRAMI (G.).

SALMOIRAGHI (A.). Guida pratica del geometra moderno. 2a edizione. Milano, tip. U. Allegretti, 1920. 8vo. 24 + 120 pp. L. 10.00

ZSIGMONDY (K.). Zum Wesen des Zahlbegriffes und der Mathematik Antrittsrede, gehalten bei der feierlichen Rektors-Inauguration am 26. Oktober 1918 in der Technischen Hochschulen zu Wien. Wien, Verlag des Verfassers, 1919.

II. ELEMENTARY MATHEMATICS.

BARKER (E. H.). Applied mathematics for junior high schools and high schools. Boston, Allyn and Bacon, 1920. 12mo. 256 pp. $1.25

CAMINATI (C.) e CAMINATI (P.). Nuovo manuale italiano logaritmico-trigonometrico con sette o dieci decimali. Fascicolo 2: Tavole di logaritmi delle funzioni circolari a sette decimali, per uso delle scuole secondarie. Piacenza, tip. V. Porta, 1919. 8vo. 50 pp. L. 3.50

CAMINATI (P.). See CAMINATI (C.).

DURELL (C. V.) and PALMER (G. W.). Elementary algebra. Part 1. London, Bell, 1920. 8 + 256 + 8 pp. 3s. 6d.

LIETZMANN (W.). Riesen und Zwerge im Zahlenreich. 2te durchgesehene und vermehrte Auflage. (Mathematisch-physikalische Bibliothek, Nr. 25.) Leipzig, Teubner, 1919. 58 pp. M. 1.50

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy and the Royal Military College, and papers in elementary engineering for the Royal Air force for the years 1910–1919. Edited by R. M. Milne. London, Macmillan, 1920. 10s. 6d.

NEUENDORFF (R.). Lehrbuch der Mathematik. Für mittlere technische Fachschulen der Maschinenindustrie. 2te Auflage. Berlin, 1919. M. 12.00

PALMER (G. W.). See DURELL (C. V.).

WIELEITNER (H.). Die sieben Rechnungsarten mit allgemeinen Zahlen. 2te, durchgesehene Auflage. (Mathematisch-physikalische Bibliothek, Nr. 7.) Leipzig, Teubner, 1920. 55 pp.

III. APPLIED MATHEMATICS.

ARRHENIUS (S.). Der Lebenslauf der Planeten. Nach der 4ten Auflage aus den Schwedischen von B. Finkelstein. Leipzig, Akadem. Verl. Gesellschaft, 1919. Geb. M. 10.00

BAILLAUD (J.). Manuel de topométrie. Opérations sur le terrain et calculs. Paris, Dunod, 1920. 4to. 8 + 222 pp. Fr. 19.50

BERGET (A.). See CHAPPUIS (J.).

BIGOURDAN (G.). Petit atlas céleste précedé d'une introduction sur les constellations. 2e édition. Paris, Gauthier-Villars, 1920. 8vo. 56 pp. + 5 maps in colors. Fr. 3.00

BLONDEL (A.). See MAUDUIT (A.).

BUSCH (F.). Beobachtung des Himmels mit einfachen Instrumenten. 2te Auflage. (Mathematisch-physikalische Bibliothek, Nr. 14.) Leipzig, Teubner, 1919. 51 pp. Geh. M. 1.00

CAUNT (G. W.). See LOUIS (H.).

CHAPPUIS (J.) and BERGET (A.). Leçons de physique générale. Tome II: Electricité et magnétisme. 3é édition, entièrement refondue par J. Chappuis et M. Lamotte. Paris, Gauthier-Villars, 1920. 8vo. 624 pp. Fr. 45.00

DELLINGER (J. H.). Principles of radio transmission and reception with antenna and coil aerials. (Scientific Paper No. 354 of the Bureau of Standards.) Washington, Government Printing Office, 1919. 8vo. 64 pp. $0.10

DRZEWIECKI (S.). Théorie générale de l'hélice. Hélices aériennes et hélices marines. Paris, Gauthier-Villars, 1920. 8vo. 12 + 184 pp. Fr. 15.00

FABER (O.). See FERRARI (—.).

FERRARI (—.). Dioptric instruments; being an elementary exposition of Gauss's theory and its applications. Translated by O. Faber. London, Harrison, 1920. 31 + 214 pp. 4s.

FINKELSTEIN (B.). See ARRHENIUS (S.).

FÖPPL (A.). Vorlesungen über technische Mechanik. 3ter Band: Festigkeitslehre. 7te Auflage. 4ter Band: Dynamik. 5te Auflage. Leipzig, 1919. M. 16.00 + 16.00

HARTING (M.). See TURRIÈRE (E.).

HEIRMAN (E.). Calculs graphiques et analytiques du béton armé. Paris, Dunod, 1920. 4to. 208 pp. Fr. 18.00

HENKEL (O.). Graphische Statik mit besonderer Berücksichtigung der Einflusslinien. 2ter Teil. Berlin, 1919. M. 1.25

LAMOTTE (M.). See CHAPPUIS (J.).

LOUIS (H.) and CAUNT (G. W.). Tacheometer tables, with an introductory chapter on tacheometric surveying. London, Longmans, 1920. 8vo. 40 + 52 pp. 12s. 6d.

MASCART (J.). See REYNAUD (P.).

MAUDUIT (A.). Electrotechnique appliquée. Machines électriques (théorie, essais et construction). Cours professé à l'Institut électrotechnique de Nancy. Avec une préface de A. Blondel. Nouveau tirage. Paris, Dunod, 1920. 8vo. 20 + 930 pp. Fr. 52.50

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The two hundred and twelfth regular meeting of the Society was held at Columbia University on Saturday, October 30, 1920, extending through the usual morning and afternoon sessions. The attendance included the following thirty-five members of the Society:

Dr. J. W. Alexander, Professor A. A. Bennett, Professor F. N. Cole, Dr. Tobias Dantzig, Dr. Jesse Douglas, Professor L. P. Eisenhart, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Mr. Philip Franklin, Dr. T. H. Gronwall, Dr. C. M. Hebbert, Dr. A. A. Himwich, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. E. A. T. Kircher, Dr. K. W. Lamson, Mr. Harry Langman, Dr. H. F. MacNeish, Professor Frank Morley, Professor W. F. Osgood, Dr. G. A. Pfeiffer, Dr. E. L. Post, Professor H. W. Reddick, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor H. W. Tyler, Professor Oswald Veblen, Professor J. H. M. Wedderburn, Mr. R. A. Wetzel, Professor H. S. White, Professor J. K. Whittemore, Dr. T. S. Yang.

President Morley occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. P. M. Batchelder, University of Texas; Miss Vevia Blair, Horace Mann School; Mr. E. H. Carus, La Salle, Ill.; Mr. W. E. Cederberg, University of Wisconsin; Mr. R. P. Conkling, Newark Technical School; Mr. P. H. Evans, Northwestern Mutual Life Insurance Company, Milwaukee, Wis.; Mr. B. L. Falconer, U. S. Civil Service Commission, Boston, Mass.; Mr. J. A. Foberg, Crane Junior College, Chicago, Ill.; Dr. Gladys E. C. Gibbens, University of Minnesota; Professor L. E. Gurney, University of the Philippines; Professor Archibald Henderson, University of North Carolina; Miss Jewell C. Hughes, University of Arkansas; Miss Claribel Kendall, University of Colorado; Mrs. M. I. Logsdon, University of Chicago; Mr. R. L. McNeal, General Motors Laboratories, Detroit, Mich.; Mr. H. L. Olson, University of Michigan; Professor Leigh Page, Yale University; Capt. H. W. Rehm, Aberdeen Proving Ground, Md.; Mr. Irwin Roman, Northwestern University; Mr.

Raleigh Schorling, Lincoln School, New York City; Mr. E. L. Thompson, Junior College, Joliet, Ill.; Dr. Bird M. Turner, University of Illinois. Four applications for membership in the Society were received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations of officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election. The Treasurer of the Society to be elected at the annual meeting was made curator of all property belonging to the Society.

It was announced that the next summer meeting of the Society will be held at Wellesley College. The Mathematical Association of America, which will hold its winter meeting at Chicago, will meet with the Society at Wellesley.

The following papers were read at the October meeting:

(1) Mr. H. S. VANDIVER: "On Kummer's memoir of 1857 concerning Fermat's last theorem. Second paper."

(2) Professor R. L. BORGER: "On total differentiability."

(3) Professor ELIZABETH LESTOURGEON: "Minima of functions of lines."

(4) Professor JOSEPH LIPKA: "Complete geometric characterization of the dynamical trajectories on a surface for any positional field of force."

(5) Professor JOSEPH LIPKA: "Complete geometric characterization of brachistochrones, catenaries, and velocity curves on a surface."

(6) Professor DUNHAM JACKSON: "On the convergence of certain polynomial approximations."

(7) Dr. J. F. RITT: "On algebraic functions which can be expressed in terms of radicals."

(8) Professor A. A. BENNETT: "The Schwarz inequality for a given symmetrical convex region and given bilinear form."

(9) Professor EDWARD KASNER: "Determination of an Einstein gravitational field by means of the paths of free particles."

(10) Professor O. E. GLENN: "An algorism for differential invariant theory."

(11) Dr. T. H. GRONWALL: "Some inequalities in the theory of functions of a complex variable."

(12) Dr. W. L. G. WILLIAMS: "Fundamental systems of formal modular seminvariants of the binary cubic."

Dr. Williams' paper was communicated to the Society through Professor L. E. Dickson. In the absence of the authors the papers of Mr. Vandiver, Professor Borger, Professor LeStourgeon, Professor Lipka, Professor Jackson, Professor Glenn and Dr. Williams were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The first paper under the present title appeared in the *Proceedings of the National Academy* for May, 1920. In it, Mr. Vandiver mentions four theorems of Kummer's, which he numbers I to IV. It is shown that Kummer's arguments for each theorem are either deficient or inaccurate. In the present paper, the work of Kummer is modified in such a way as to yield proofs of theorems I and IV.

2. Professor Borger's paper contains a necessary and sufficient condition that a function of two independent variables may possess a total differential in any point where its partial derivatives exist and are finite. The extension to n variables is also indicated.

For a function of two variables this condition may be stated as follows: If the limits

$$\lim_{(k,\, h \doteqdot 0)} \frac{f(x+h, y+k) - f(x, y+k)}{h},$$

$$\lim_{(h,\, k \doteqdot 0)} \frac{f(x+h, y+k) - f(x+h, y)}{k}$$

exist; and if

$$\lim_{(k,\, h \doteqdot 0)} \frac{f(x+h, y+k) - f(x, y+k)}{h} = f_x,$$

$$\lim_{(h,\, k \doteqdot 0)} \frac{f(x+h, y+k) - f(x+h, y)}{k} = f_y,$$

the function $f(x, y)$ is totally differentiable in the point (x, y) and conversely.

The notation $\lim_{(h,\, k \doteqdot 0)}$ signifies that h and k approach zero so that the absolute value of the first variable in the parenthesis is less than or equal to that of the second.

3. Fréchet has defined first and second differentials for a function of a line or functional $F(\lambda)$. If $\lambda_0(x) + \eta(x)$ is a variation of $\lambda_0(x)$, the first differential is a linear functional $L(\eta)$ and the second is expressible in the form $B(\eta, \eta)$, where $B(u, v)$ is a bilinear functional in the independent arguments $u(x)$ and $v(x)$. The definitions of Fréchet apply only to functionals and differentials having continuity of order zero. Professor Le Stourgeon shows that if the continuity is of the first order, as in the case of the integrals of the calculus of variations, the differentials $L(\eta)$ and $B(\eta, \eta)$ are expressible in the forms

$$L(\eta) = \int_a^b \eta(x)du(x) + \int_a^b \eta'(x)du_1(x),$$

$$B(\eta, \eta) = \int_a^b\int_a^b \eta(x)\eta(y)d_{xy}p(x, y) + 2\int_a^b\int_a^b \eta(x)\eta'(y)d_{xy}q(x,y) + \int_a^b\int_a^b \eta'(x)\eta'(y)d_{xy}r(x, y).$$

If the functional $F(\lambda)$ has a minimum at λ_0, then it is proved that u and u_1 must satisfy an equation of the form

$$u_1(x) - \int_a^x u(x)dx = kx + l,$$

where k and l are constants. Furthermore, under certain restrictions, a necessary condition for a minimum analogous to the Jacobi condition of the calculus of variations is deduced. It is proved that when $F(\lambda_0)$ is a minimum the equation

$$\int_a^b [u(y)d_yq(y, x) + u'(y)d_yr(x, y)] - \int_a^x\int_a^b [u(y)d_yp(x, y) + u'(y)d_yq(x, y)]dx = kx + l$$

can have no solution $u(x)$, except $u(x) \equiv 0$, vanishing at $x = a$ and a point $x = x'$ between a and b.

The paper appeared in the October issue of the *Transactions*.

4. In this paper, Professor Lipka derives five geometric properties of the system of trajectories generated by the motion of a particle on any constraining surface under any

positional forces. The properties involve the system of trajectories, certain plane systems associated with them by projection into the tangent planes, the lines of force, and isothermal nets of curves, and are stated in terms of geodesic curvature, osculating and hyperosculating geodesic circles, and osculating parabolas. It is shown that the five geometric properties completely characterize the system of trajectories in the sense that any triply infinite system of curves on a surface which possesses these five properties may be considered as generated by the motion of a particle in a unique field of force.

5. This paper presents a study (analogous to that made for dynamical trajectories under positional forces) of certain systems of curves on a surface termed "n" systems—which include brachistochrones, catenaries, velocity curves, and dynamical trajectories as special cases. The field of force is conservative. Professor Lipka derives five geometric properties and shows that these completely characterize an "n" system, in the sense that any triply infinite system of curves on a surface which possesses these five properties may be considered as an "n" system—system of dynamical trajectories, brachistochrones, catenaries, velocity curves, etc., depending on the value of "n."

6. In this paper, Professor Jackson extends to a problem of polynomial approximation the method of treatment used in a paper presented to the Society at the summer meeting in Chicago, on the corresponding trigonometric case. It is assumed that $f(x)$ is a given function, continuous for $a \leqq x \leqq b$, and that $P_{mn}(x)$ is the polynomial of the nth degree which gives the best approximation to $f(x)$ over the interval, in the sense of the integral of the mth power of the absolute value of the error. The conclusions reached are not exactly parallel to those in the trigonometric case. It is found that, when m is held fast and n is allowed to become infinite, $P_{mn}(x)$ converges uniformly to the value of $f(x)$ for $a \leqq x \leqq b$, provided that $\lim_{\delta=0} \omega(\delta)/\sqrt[m]{\delta^2} = 0$, where $\omega(\delta)$ is the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. The condition as stated is significant only for $m > 2$; its form is appropriately modified for $1 \leqq m \leqq 2$. Somewhat less stringent conditions are found for convergence in the interior of the interval.

7. Let $F(w, z) = 0$ be an irreducible algebraic equation of prime degree n in w. Let the number of values of z for which the Riemann surface for w has branch points be q. Dr. Ritt shows that if w can be expressed in terms of radicals, then

$$q \leqq 4 + \frac{4p}{n-1},$$

where p is the genus of the given algebraic relation.

It can be shown from this inequality that for every genus greater than zero, there is an upper bound to those prime numbers n which can serve as the degree of w in an algebraic relation which permits w to be expressed in terms of radicals.

For the special and interesting case of $p = 0$, we have $q = 2$, 3 or 4. If the monodromy group contains a cyclic substitution, we must have $q = 2$ or $q = 3$. In that case the Riemann surface for w can be changed by a linear transformation either into the surface for $z^{1/n}$ or into the surface for $\cos((\cos^{-1} z)/n)$. It follows that the only polynomials of prime degree whose inverses can be expressed in terms of radicals are $(az + b)^n + c$ and $cf_n(az + b) + d$, where $\cos nz = f_n(\cos z)$.

For $q = 4$, the equations are those which appear in the transformations of prime order of the elliptic functions.

If $q = 3$ and the monodromy group contains no cyclic substitution, we must have $n = 3r + 1$, $4r + 1$ or $6r + 1$, to each of which cases corresponds a special class of equations.

8. Starting with an arbitrary symmetrical convex region, in a domain of a linear set of elements, Professor Bennett constructs a norm, having a linearly homogeneous property, and satisfying a "triangle inequality." For a dual set of variables, an inner product and a dual norm are obtained. These will in every case satisfy the Schwarz inequality familiar in integral equations. The norms thus obtained are determined completely by the character of the given symmetrical convex region.

9. Professor Kasner shows that a four-dimensional manifold obeying Einstein's gravitational equations is essentially determined by its geodesics (paths of particles). Two Einstein manifolds cannot admit geodesic representation without being equivalent (applicable). Hence a complete knowledge of the

orbits of planets in the solar gravitational field determines that field completely, and therefore makes it possible to predict the shape of the light rays. The converse determination, by light rays alone, was discussed in one of the "Five notes on relativity" read at the summer meeting in Chicago. Thus we have in particular a two-fold connection between the motion of the perihelion of Mercury (43″), and the deflection of a light ray (1.7″).

10. The initial point of view in Professor Glenn's paper is obtained from the poles of the transformation on the differentials which one derives from relations $x_r = x_r(y_1, y_2)$ $(r = 1, 2)$ by which a differential quantic f is transformed. These are zeros of linear differential covariants $df_{\pm 1}$ appertaining to a domain R defined by certain irrational expressions in the functions x_r and their derivatives. The coefficients of the expansion of f in terms of $df_{\pm 1}$ as arguments are relative differential parameters, here called invariant elements, appertaining to R. Moreover every differential parameter (or covariant) is, in R, a function of the invariant elements and their derivatives, which accordingly afford a basis of classification for the known types of parameters and for several new types belonging to various domains. The methods of the paper give a direct approach to finiteness theorems and proofs in the subject. A complete system of 31 parameters of a so-called orthogonal type for the differential quantic of order six is produced in the paper, and, in addition, certain more general systems of an extended orthogonal type.

11. In this paper, Dr. Gronwall establishes inequalities analogous to that of Carathéodory for the cases where the real or imaginary part, or both, of a power series convergent in the unit circle is bounded above or below, or both ways.

12. Dr. Williams proves certain general theorems regarding formal modular seminvariants of the binary cubic, modulo p, a prime. A method of deriving a fundamental system of formal seminvariants for any particular prime is outlined and the method is applied to the cases $p = 5$ and $p = 7$. The results for $p = 5$ agree with those found by L. E. Dickson and published in his Madison Colloquium Lectures.

F. N. Cole,
Secretary.

THE MATHEMATICAL CONGRESS AT STRASBOURG.

A MATHEMATICAL congress, called as a result of the general recommendations of the Brussels conference, was held at the University of Strasbourg from September 22 to September 30, 1920. Although, as would naturally be expected, it was attended chiefly by French mathematicians, there was a fair attendance from other allied countries. There were some twelve members from the United States and six from England, but at the time of preparing this report there had appeared no printed list of those present.

In addition to the meetings of the several sections, the general programme was as follows:

Wednesday, September 22: Official opening of the congress, under the presidency of M. Alapetite, Commissaire général of the republic; address of welcome, with responses by representatives of various countries. Professor L. E. Dickson responded for this country. Visit to the university. First general session: election of president, vice-presidents, and secretary. Professor Picard was chosen president by unanimous vote. Reception to members of the congress, held in the Salle des Fêtes of the university.

Thursday, September 23: Visits to the four museums of the city. General session: address by Sir Joseph Larmor on the nature of the ether. Reception by the Society of friends of the university.

Friday, September 24: Visits to points of interest in the city. General session: address by Professor L. E. Dickson. Reception at the Hôtel-de-Ville. General meeting in honor of the congress, organized by the Société des Sciences du Bas-Rhin: address by General Tauflieb on science in Alsace; concert.

Saturday, September 25: General session: address by Professor C. J. de la Vallée Poussin. Reception at the Commissariat général.

Sunday-Monday, September 26–27: Excursions to Saint-Odile and on the Rhine.

Tuesday, September 28: General sessions: addresses by Professors Vito Volterra and N. E. Nörlund. Closing session. Banquet.

Wednesday–Thursday, September 29–30: Excursions to Saverne and Linge.

At the section meetings, held September 23–25, 27–28, the following papers were read:

SECTION I. ARITHMETIC, ALGEBRA, ANALYSIS.

YOUNG: Sur la définition de l'aire et du volume.

DICKSON: Homogeneous polynomials with a multiplication theorem.

CHÂTELET: La loi de réciprocité et les corps Abéliens.

DANIELL: On Stieltjes integrals and Volterra composition.

AMSLER: Sur le calcul symbolique sommatoire.

FUETER: Einige Sätze aus der Theorie der complexen Multiplikation der elliptischen Funktionen.

DENJOY: Sur une classe d'ensembles parfaits en relation avec les fonctions admettant une dérivée généralisée.

STOÏLOW: Sur les ensembles de mesure nulle.

DU PASQUIER: Sur une théorie des nombres complexes.

WIENER: On certain iterative properties of bilinear operations.

DRACH: L'intégration logique des équations différentielles; applications à l'analyse.

HADAMARD: Sur la solution élémentaire des équations linéaires aux dérivées partielles et sur les propriétés des géodésiques.

TAKAGI: Sur quelques théorèmes généraux de la théorie des nombres algébriques.

REY PASTOR: Sur la transformation conforme.

TYPPA: Sur les équations du troisième degré.

STÖRMER: Méthode d'intégration numérique des équations différentielles.

RÉMOUNDOS: Sur le module et les zéros des fonctions analytiques.

VAROPOULOS: Sur le module maximum des fonctions algébroïdes.

RIABOUCHINSKY: Sur le calcul des valeurs absolues.

ZERVOS: Remarques sur certaines transformations des équations aux dérivées partielles.

RADL: Sur la transformation des équations différentielles linéaires.

BOUTROUX: Sur une équation différentielle et sur une famille de fonctions entières.

LEFSCHETZ: Quelques remarques sur la multiplication complexe.

WAWRE: Sur un système d'équations à une infinité d'inconnues.

WIENER: On the theory of sets of points in terms of continuous transformations.

DE RUYTS: Une propriété simple des systèmes transformables.

OGURA: Sur la théorie de l'interpolation.

VALIRON: Sur quelques points de la théorie des fonctions entières.

ZERVOS: Sur l'intégration de certains systèmes différentiels indéterminés.

WALSH: On the location of the roots of polynomials.

ZAREMBA: Sur un théorème fondamental relatif à l'équation de Fourier.

YOUNG: Sur certaines intégrales doubles.

SAKELLARIOU: Sur les solutions discontinues du problème du calcul des variations dans l'espace à n dimensions.

SECTION II. GEOMETRY.

APRILE: Le congruenze di coniche.

BYDZOWSKY: Sur les transformations quadratiques reproduisant une quartique elliptique plane analytique.

TAYLOR: La géométrie des variables complexes.

CARTAN: Sur le problème général de la déformation.

DRACH: L'intégration logique des équations différentielles; application à la géométrie et à la mécanique.

EISENHART: Transformation des systèmes conjugués R.

SOBOTKA: Sur la deuxième indicatrice en un point d'une surface.

HOSTINSKY: Sur les propriétés de la sphère qui touche quatre plans tangents consécutifs d'une développable.

CLAPIER: Sur la transformation de Lie.

LE ROUX: Sur la géométrie des déformations des milieux continus.

EISENHART: Transformation des surfaces applicables sur une quadrique.

MURRAY: Method of classifying all polygons having a given set of vertices.

SECTION III. MECHANICS, APPLIED MATHEMATICS.

VANDERLINDEN: Les théories d'Einstein et leurs applications à l'astronomie.

BRILLOUIN: Sur un type d'action à hérédité discontinue et les équations différentielles des mouvements qui en résultent.

SCHWOERER: Détermination de l'équation séculaire de la terre dans la théorie d'Arrhenius.

GUILLAUME: Expression mono- et poly paramétrique du temps dans la théorie de la relativité.

WILIGENS: Représentation géométrique du temps dans la théorie de la relativité.

HADAMARD: Sur le problème mixte pour une équation linéaire aux dérivées partielles.

BANERJI: Some problems in earthquakes.

BOCCARDI: Sur le déplacement du pôle.

DA COSTA-LOBO: Sur la courbe décrite par le pôle sur la surface de la terre.

BOCCARDI: Sur les approximations numériques et les sciences d'observation.

FARID-BOULAD: Nouveau théorème pour calculer les tensions des barres surabondantes des poutres et arcs à montants et croix de Saint André.

GREENHILL: La fonction potentielle uniaxiale et sa fonction de force orthogonale.

HATZIDAKIS: Sur quelques formules de géométrie cinématique.

HOSTINSKY: Sur un problème général de la mécanique vibratoire.

MAILLARD: Mise au point des hypothèses cosmogéniques-nébulaires.

ROSENBLATT: Sur la théorie des figures d'équilibre des masses fluides animées d'un mouvement de rotation.

RIABOUCHINSKY: Sur la résistance des fluides.

LARMOR: Sur les pressions des ondes sonores.

GULDENBERG: Une application des polynômes d'Hermite à un problème statistique.

DE DONDER: Sur la gravifique.

LARMOR: Sur les rayons diffractés attachés aux images optiques.

BARRAU: Sur la cinématique plane.

BAUER: Remarques élémentaires sur le principe de relativité en electrodynamique.

SECTION IV. PHILOSOPHY, HISTORY, PEDAGOGY.

GÉRARDIN: Décomposition des nombres. Machines à congruences. Des nombres entiers. Jeux scientifiques.

BROCARD: 24 propositions de Fermat.
DELAPORTE: Sur la réforme du calendrier.
DU PASQUIER: Sur les nombres transfinis.
GREENHILL: Les fonctions de Fourier et Bessel comparées.
D'OCAGNE: La pratique courante de la méthode nomographique des points alignés: à propos de ses applications de guerre.
GROSSMANN: Sur l'état de publication des œuvres d'Euler.
POSTIGLIONE: Cyclómetrie mécanique.
DUBECQ: Communication sur l'enseignement, République Argentine.
ZERVOS: Sur l'enseignement mathématique.

DAVID EUGENE SMITH.

NOTE ON A METHOD OF PROOF IN THE THEORY OF FOURIER'S SERIES.

BY PROFESSOR DUNHAM JACKSON.

(Read before the American Mathematical Society September 7, 1920.)

IT has been pointed out on various occasions* that if $f(x)$ is a continuous function of period 2π satisfying the Lipschitz-Dini condition, that is, if $\lim_{\delta=0} \omega(\delta) \log \delta = 0$, where $\omega(\delta)$ is the maximum of the oscillation of $f(x)$ in an interval of length δ, then the uniform convergence of the Fourier series for $f(x)$ can be inferred almost immediately from the following two propositions:

A.† If $f(x)$ satisfies the Lipschitz-Dini condition,‡ there exists for every positive integral value of n a finite trigonometric sum $\tau_n(x)$, of order n at most, such that $\lim_{n=\infty} r_n \log n = 0$, where r_n is the maximum of $|f(x) - \tau_n(x)|$.

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* Cf., e.g., Lebesgue, "Sur les intégrales singulières," *Annales de la Faculté de Toulouse*, series 3, vol. 1 (1909), pp. 25–117; pp. 116–117.

† Cf., e.g., Lebesgue, loc. cit., p. 116; D. Jackson, "On the approximate representation of an indefinite integral, etc.," *Transactions Amer. Math. Society*, vol. 14 (1913), pp. 343–364; p. 350.

‡ It is understood throughout the paper that every function considered has the period 2π.

*B.** If $\varphi(x)$ is any continuous† function, and $S_n(x)$ the partial sum of the Fourier series for $\varphi(x)$ to terms of order n, then $|S_n(x)|$ can not exceed $KM \log n$, where M is the maximum of $|\varphi(x)|$, and K is an absolute constant.

The central point in the proof is the fact that $\tau_n(x)$ is identical with the partial sum of its own Fourier series to terms of order n. It is the purpose of this note to show that similar reasoning can be applied to the arithmetical mean of Fejér,‡ in spite of the fact that the Fejér mean formed for a finite trigonometric sum $\tau_n(x)$ is not the same as $\tau_n(x)$. It is necessary to change the argument somewhat, but there is no difficulty in making the required modification.

Let $f(x)$ be now an arbitrary continuous function, and let $\sigma_n(x)$ be the arithmetical mean of the partial sums of the Fourier series for $f(x)$, to terms of order n. The uniform convergence of $\sigma_n(x)$ to the value $f(x)$ is to be deduced from the propositions:

C. (Weierstrass's theorem.)§ If $f(x)$ is continuous, there exists for every positive integral value of n a finite trigonometric sum $\tau_n(x)$, of order n at most, such that $\lim_{n=\infty} r_n = 0$, where r_n is the maximum of $|f(x) - \tau_n(x)|$.

D.‖ If $\varphi(x)$ is any continuous function (more generally, any integrable function), and $\sigma_n(x)$ the Fejér mean of the Fourier series for $\varphi(x)$ to terms of order n, then $|\sigma_n(x)|$ can not exceed M, where M is the maximum of $|\varphi(x)|$.

Let ϵ be any positive quantity. Let a finite trigonometric sum $\tau_p(x)$, of order p, be determined, according to Proposition C, so that

$$|f(x) - \tau_p(x)| < \tfrac{1}{3}\epsilon.$$

Let $\sigma_{n1}(x)$ be the Fejér mean, of order n, for the function $\tau_p(x)$, and $\sigma_{n2}(x)$ the corresponding mean for the function $f(x) - \tau_p(x)$.

* Cf., e.g., Lebesgue, loc. cit., p. 116; D. Jackson, "On approximation by trigonometric sums and polynomials," *Transactions Amer. Math. Society*, vol. 13 (1912), pp. 491–515; pp. 502, 512–515.

† It is sufficient for the truth of the statement that φ be integrable, but for present purposes there is no need of speaking of any but continuous functions.

‡ Fejér, "Untersuchungen über Fouriersche Reihen," *Mathematische Annalen*, vol. 58 (1904), pp. 51–69.

§ Weierstrass, "Ueber die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen," *Berliner Sitzungsberichte*, 1885, pp. 633–639, 789–805; p. 801.

‖ Fejér, loc. cit., p. 60.

Then

$$\sigma_n(x) = \sigma_{n1}(x) + \sigma_{n2}(x). \tag{1}$$

By Proposition D,

$$|\sigma_{n2}(x)| < \tfrac{1}{3}\epsilon,$$

$$|f(x) - \tau_p(x) - \sigma_{n2}(x)| < \tfrac{2}{3}\epsilon, \tag{2}$$

for all values of n. The quantity $\sigma_{n1}(x)$ is the arithmetical mean of $n + 1$ finite trigonometric sums, of which all from the $(p + 1)$th on, if $n \geqq p$, are identical with $\tau_p(x)$, while each of the first p is composed of a part of the terms of $\tau_p(x)$. Added together, the first p sums which enter into the mean give a finite trigonometric sum $\omega_{p-1}(x)$, which is of order $p - 1$ at most, and independent of n. So $\sigma_{n1}(x)$ can be written in the form

$$\sigma_{n1}(x) = \frac{\omega_{p-1}(x) + (n + 1 - p)\tau_p(x)}{n + 1}$$

$$= \tau_p(x) + \frac{\omega_{p-1}(x) - p\tau_p(x)}{n + 1}.$$

As the last numerator is independent of n, $\sigma_{n1}(x)$ approaches $\tau_p(x)$ uniformly as n becomes infinite—a fact which is fairly obvious in the first place—and, if n is sufficiently large,

$$|\tau_p(x) - \sigma_{n1}(x)| < \tfrac{1}{3}\epsilon. \tag{3}$$

By combination of (1), (2), and (3), for values of n satisfying (3),

$$|f(x) - \sigma_n(x)| < \epsilon,$$

which completes the convergence proof.

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ON IMPLICIT FUNCTIONS.

BY MR. F. H. MURRAY.

METHODS of solving a system of m equations in $n > m$ variables, by reducing the problem to that of solving a set of differential equations, have already been given;* the aim of this paper is to present a method by which the system of m equations can be reduced immediately to a system of m differential equations of the first order, from which reduction the existence theorem and a method of constructing the solutions follow directly. For simplicity the case of two equations in four variables will be treated.

§ 1. *Reduction to a System of Differential Equations.*

It is required to solve the system

$$(1) \qquad f_1(x, y, u, v) = 0, \quad f_2(x, y, u, v) = 0$$

in the neighborhood of a set of values (x_0, y_0, u_0, v_0) for which

$$f_1(x_0, y_0, u_0, v_0) = 0, \quad f_2(x_0, y_0, u_0, v_0) = 0.$$

There is no loss in generality in assuming $x_0 = y_0 = u_0 = v_0 = 0$; suppose x, y to be the independent, u, v the dependent variables. It will be assumed that all the first partial derivatives exist and are continuous in the neighborhood of the origin defined by the inequalities

$$(R) \qquad \sqrt{x^2 + y^2} \leqq a, \quad |u| \leqq b, \quad |v| \leqq b.$$

Also, assume that the Jacobian

$$\Delta = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$$

does not vanish at the origin (0, 0, 0, 0); consequently the constants a, b can be so chosen that $\Delta \neq 0$ in R. Introduce

* Horn, Gewöhnliche Differentialgleichungen beliebiger Ordnung.

polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Equations (1) become

$$(1') \quad f_1(r\cos\theta, r\sin\theta, u, v) = 0, \quad f_2(r\cos\theta, r\sin\theta, u, v) = 0.$$

If functions $u(r, \theta)$, $v(r, \theta)$, satisfying (1′) can be found, such that $\partial u/\partial r$, $\partial v/\partial r$ are defined throughout the region R, we must have

$$(2) \quad \begin{aligned} \frac{\partial f_1}{\partial x}\cos\theta + \frac{\partial f_1}{\partial y}\sin\theta + \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial r} &= 0, \\ \frac{\partial f_2}{\partial x}\cos\theta + \frac{\partial f_2}{\partial y}\sin\theta + \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial r} &= 0. \end{aligned}$$

Since $\Delta \neq 0$,

$$(3) \quad \begin{aligned} \frac{\partial u}{\partial r} &= \frac{\begin{vmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial x} \end{vmatrix}\cos\theta + \begin{vmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial y} \end{vmatrix}\sin\theta}{\Delta}, \quad u(0) = 0, \\ \frac{\partial v}{\partial r} &= \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial u} \end{vmatrix}\cos\theta + \begin{vmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} \end{vmatrix}\sin\theta}{\Delta}, \quad v(0) = 0. \end{aligned}$$

In these equations consider θ as a parameter, u, v, r as the variables; equations (3) can be written in the form

$$(3') \quad \frac{du}{dr} = A(u, v, r), \quad \frac{dv}{dr} = B(u, v, r), \quad u(0) = v(0) = 0.$$

Suppose the original functions f_1, f_2 such that for every θ, $0 < \theta \leqq 2\pi$, there exist constants C, D if u', u'', v', v'', are in R, such that

$$(4) \quad \begin{aligned} |A(u'', v'', r) - A(u', v', r)| &< C|u'' - u'| + D|v'' - v'|, \\ |B(u'', v'', r) - B(u', v', r)| &< C|u'' - u'| + D|v'' - v'|. \end{aligned}$$

§ 2. *Final Solution.*

Suppose that for $u = v = 0$, $r \leqq h$, $|A| \leqq M_0$, $|B| \leqq M_0$; for u, v, r in R, $|A| \leqq M$, $|B| \leqq M$. Then from the existence theorem for ordinary differential equations,* if $C + D = K$, the equations (3′) have solutions u, v if $0 \leqq r \leqq \delta$, where δ is the larger of the quantities δ_1, δ_2,

$$\delta_1 \leqq a \leqq \frac{b}{M}, \quad \delta_2 \leqq a \leqq \frac{1}{K}\log\left(1 + \frac{bK}{M_0}\right).$$

The functions u, v satisfy (3′), hence have derivatives with respect to r.

Since A, B are periodic in θ, the constants C, D have upper limits independent of θ, consequently K, M_0 can be defined independent of θ; the series defining u, v converge uniformly in θ, hence u, v are continuous functions of θ. The constant M can also be defined independent of θ, with the result that the solutions u, v are continuous functions of r, θ, and differentiable with respect to r, if $0 \leqq r \leqq \delta$.

Introduce these functions in (1′); since

$$\frac{\partial f_1}{\partial r} = \frac{\partial f_2}{\partial r} = 0, \quad f_1 = \phi_1(\theta), \quad f_2 = \phi_2(\theta).$$

But within the region $0 \leqq r \leqq \delta$, f_1, f_2 are continuous functions of r, θ; for $r = 0$, $x = y = u = v = 0$, consequently

$$\phi_1(\theta) = 0, \quad \phi_2(\theta) = 0.$$

These equations must hold for every θ, hence are identities, and equations (1′) are identically satisfied. The functions u, v are the solutions required.

For n variables $x_1, x_2, \cdots, x_n$,

$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}, \qquad \cos\theta_i = x_i/r.$$

* Picard, Traité d'Analyse, 2, pp. 343, 345.

ON A PENCIL OF NODAL CUBICS. SECOND PAPER.

BY PROFESSOR NATHAN ALTSHILLER-COURT.

(Read before the American Mathematical Society April 24, 1920.)

1. REFERRING to the notations of my first communication,* let Γ_n, Γ_s be two cubics of the pencil Γ, and let X_n, X_s, F be the points of intersection of a fixed line l through the double point O with Γ_n, Γ_s, and the basic line ABC respectively; let X_n', X_s', F' be the analogous points on any other line l' through O. The lines X_nX_n', X_sX_s', meet on ABC according to proposition 8 of the paper referred to, and therefore

$$(OFX_nX_s) = (OF'X_n'X_s').$$

Thus a variable line l' through the double point O meets the two cubics Γ_n, Γ_s in two points X_n', X_s' which, with the double point O and the trace F' of l' on ABC, form an anharmonic ratio having a constant value. Consequently:†

Two nodal cubics having in common three collinear points, the double point, and the tangents at this point, are homological.

The double point and the base of the three collinear points are respectively the center and the axis of homology.

Two tricuspidal quartics having in common three concurrent tangents, the double tangent, and the two points of contact with this line, are homological.

The double tangent and the common point of the three concurrent tangents are respectively the axis and the center of homology.

2. Any two cubics of the pencil Γ are thus homological, i.e., Γ is a pencil of homological cubics. This fundamental property of Γ furnishes immediately a second proof of most of the propositions of my first paper. It also brings to light many new propositions. For instance:

* This BULLETIN, vol. 26, p. 203 (February, 1920).

† M. Chasles, Traité de Géométrie supérieure, 2d edition, p. 350.

(*a*) The triangle formed by the three inflexional tangents of any cubic of the pencil Γ is homological to the corresponding triangle of any other cubic of the pencil, the double point O being the center of homology, hence: *The vertices of the triangle formed by the inflexional tangents of a variable cubic of the pencil* Γ *describe three straight lines passing through the double point.*

(*b*) It is known that the lines joining the pairs of corresponding points on a nodal cubic Γ_n envelop a conic ω_n, which is the Cayleyan of the cubic of which Γ_n is the Hessian. The conics (ω) corresponding to ω_n in the various curves of the pencil Γ are homological to ω_n. Hence: *The conics* (ω) *of the cubics of the pencil* Γ *have two points in common on the basic line of the pencil.*

It should be noticed however that the conics (ω) do not, in general, form a pencil. Being homological, in order to form a pencil, they would have to be tangent to the lines projecting from the center of homology O the two points that the conics have in common with the axis of homology ABC* at these points. Now it is known that the tangents from O to ω_n are the tangents OT_1, OT_2 to Γ_n at O, their points of contact with ω_n being the points determined on OT_1, OT_2 by the inflectional line i_n of Γ_n.

3. In the cases when the common elements of the two curves of section 1 are of some special nature or are taken in some special position the mutual relation of the two curves may become of particular interest. For lack of space we shall give only one example.

The tricuspidal hypocycloid (or tricusp) is a quartic touching the line at infinity at the cyclical points, according to a proposition due to Cremona.† If we consider two such curves having three concurrent tangents in common, their axis of homology will be at infinity. Consequently: *Two tricuspidal hypocycloids having three concurrent tangents in common are similar and similarly placed. The point of intersection of their common tangents is the center of similitude of the two curves.*

University of Oklahoma,
February, 1920.

* C. Servais, "Sur les faisceaux de coniques," sec. 9, *Le Matematiche pure ed applicate*, vol. I (December, 1901).

† See R. C. Archibald, "The cardioid and the quartics with three cusps," *Annals of Mathematics*, 2d series, vol. 4 (1903).

HAUSDORFF'S GRUNDZÜGE DER MENGENLEHRE.

Grundzüge der Mengenlehre. By FELIX HAUSDORFF. Veit and Company, Leipzig, 1914. viii + 476 pp.

IF there are still mathematicians who hold the theory of aggregates under general suspicion, and are reluctant to grant it full recognition as a rigorous, mathematical discipline, they will find it hard to retain their doubts under fire of the logic of Hausdorff's treatise. It would be difficult to name a volume in any field of mathematics, even in the unclouded domain of number theory, that surpasses the Grundzüge in clearness and precision.

But it is only in a subsidiary rôle that the Grundzüge is an answer to the skeptics. Its most striking feature is that it is a work of art of a master. No one thoroughly acquainted with its contents could fail to withhold admiration for the happy choice and arrangement of subject matter, the careful diction, the smooth, vigorous and concise literary style, and the adaptable notation; above all things, however, for the highly pleasing unifications and generalizations and the harmonious weaving of numerous original results into the texture of the whole.

It is not an uncommon fault of authors of treatises on general subjects to expound their own researches with an unwarranted degree of detail; so that at times, if one has no other evidence, one may be rightly led to suspect particular portions, on account of their remoteness from the central ideas, of being the author's own handiwork. This fault is not shared by the Grundzüge. Few treatises on as comprehensive a subject as the theory of aggregates contain as large a proportion of the author's investigations; yet the parts that are distinctly Hausdorff's own contributions are properly inserted in view of their generality and in relation to other topics.

The author is endowed with a keen psychological and didactic instinct that prompts him to depart from his usual succinctness when engaged in the clarification of the more important ideas. The following quotation may serve to illustrate this pedagogic sense and the general lucidity of style, not without an occasional glimmer of humor. After speaking

of the primitive stage of numerical comparison—by means of direct, successive mating—of a pile of apples and a pile of pears, he says (page 45): "But if the apples and the pears are in different places, and the transportation of one pile to the other is attended with difficulties, the inventive mind of man will in the next stage make use of an intermediary set of conveniently transportable objects, such as stones, shells, or chips, and infer the equivalence $A \sim B$ from the equivalences $A \sim C$, $B \sim C$. Finally, however, even this earthly residuum will be eliminated, and the intermediary set will be replaced by a system of spoken, written, or thought symbols, —the number symbols, 1, 2, $\cdots$. *Comparison* turns into *counting*, and equivalent sets now acquire a common property, the number of their elements.

"These remarks, for which no claim whatsoever is entered on psychological or cultural-historical grounds, are intended merely to make clear that equivalence is the natural foundation for the comparison of aggregates, and that by its means we may undertake even the seeming paradox of counting infinite sets."

That the author enjoyed himself while at work may be seen from such passages as the following (page 61): "From an 'alphabet,' i.e. a finite set of 'letters,' we may construct a countable assemblage of finite complexes [= ordered sets] of letters, i.e. 'words,' among which, of course, meaningless words such as abracadabra occur. If in addition to the letters, other elements are used, such as punctuation marks, type-spacings, numerals, notes, etc., we see that the assemblage of all books, catalogs, symphonies and operas is also countable, and would remain countable even if we were to employ a countable set of symbols (but for each complex only a finite number). On the other hand, if in the case of a finite number of symbols we restrict the complexes to a maximum number of elements, agreeing for example, to rule out words of more than one hundred letters and books of more than one million words, these assemblages become finite; and if we assume with Giordano Bruno an infinite number of heavenly bodies, with speaking, writing, and musical inhabitants, it follows as a mathematical certainty that on an infinite number of these heavenly bodies there will be produced the same opera with the same libretto, the same names of the composer, the author of the text, the members of the orchestra and the singers."

Several other examples of spirited and colorful language are the following. Speaking of the equivalence of the whole and the part of an infinite set, he says (page 34): "Of course, when asserted in the rather provocative form, A has as many elements as B, it is one of those 'paradoxes of the infinite' that shock the unprepared mind." Again, page 34: "A segment and an arbitrarily small partial segment, a kilometer and a millimeter, the sun's globe and a drop of water have in this sense the 'same number of points.'" Page 48: ". . . we shall have to desist from giving every proper subset a cardinal number $< \mathfrak{a}$; we must violate the hallowed axiom 'totum parte majus,' as we must in general expect that calculation with infinite cardinals will deviate in many respects from that with finite cardinals, without thereby espying the minutest objection against these infinite numbers." On page 63, after learning that $10^{\aleph_0} = \aleph$, we are led to the equations $\aleph = 2^{\aleph_0} = 3^{\aleph_0} = \cdots$ by the remark that "the fact that we have ten fingers is obviously without influence on the theory of aggregates." Page 60: "The equivalence of the set of whole numbers with the much more inclusive set of rational numbers belongs to those facts of the theory of aggregates which impress you on your first acquaintance with them as astonishing and even paradoxical; especially if you have before your eyes the geometric representation (of the correspondence between the numbers and the points of a straight line), and picture to yourself on the one hand, the 'integral' points, which lie isolated at finite distances from one another, and on the other hand, the 'rational' points, which are distributed over the entire line as dust of more than microscopic fineness."

One of the characteristic traits of the style is its continuity, brought about by the neat and confident conjunctional devices of the author. On page 335 there is need of the awful descent from the general spaces previously considered to the very special euclidean plane. The author is unwilling to take the plunge without assuring himself of the reader's good-will: "In the subsequent discussion we prefer to confine ourselves to the plane. The extension to three or more dimensional space presents in part not inconsiderable difficulties, because the rôle which polygons play in the plane falls there to the lot of the much less simple polyedra and hyper-polyedra. Even in the plane we shall find that the apparently most

plausible, intuitive assertions require fairly complicated proofs. A certain prolixity is already produced by the fact that sets that are not compact, and are therefore unbounded, behave in many respects not like bounded sets, and indeed, less regularly. A radical remedy would be the adjunction of a point 'at infinity,' as in function theory; but thereby the character of a metrical space is destroyed, and if again, you get rid of this evil by means of stereographic projection on the sphere, you lose in exchange certain elementary geometric advantages of the plane. We must, therefore, come to terms as best we can with the 'dreary infinities of homaloidal space' as Clifford says."

We now turn to a more detailed description of the contents. In view of the great wealth of ideas—made possible by the concentrated style—we must refrain from attempting to discuss even all the important topics; only some of these can we describe in detail, and we shall give preference to the more novel or the less technical features.

The book is fittingly inscribed to Georg Cantor, "the creator of the theory of aggregates."

Chapter I (pages 1–31) deals with the sum (denoted by $\mathfrak{S}$), section (Durchschnitt, denoted by $\mathfrak{D}$), and difference of sets. Of the numerous topics treated in this chapter, we shall discuss the notion of aggregate, the principle of duality, and symmetric sets. Among the other topics may be mentioned difference chains, rings and fields (Ringe und Körper), sequences of sets and of real numbers, σ- and δ-systems, and the non-convergence points of a sequence of functions.

Aggregate is defined in the cantorean naïve fashion—as distinguished from the less debatable, but more restraining manner of Zermelo—as a whole constituted by the conceived assembling of individuals. The author makes it clear that it is inadvisable on pedagogical and other grounds to found everything upon Zermelo's Grundlagen; paradoxes are duly banished, however, by appropriate interpretation of the naïve definition (see page 106 for the disposal of the Burali-Forti antinomy).

If $A_1, A_2, \cdots$ are subsets of a set M and $\bar{A}_1 = M - A_1$, $\bar{A}_2 = M - A_2, \cdots$ their complements,—we denote generally by $\bar{X}$ the complement of X in M—, then the complement in M of the sum of the given sets is the section of their com-

plements $[\overline{\mathfrak{S}(A_1, A_2, \cdots)} = \mathfrak{D}(\overline{A}_1, \overline{A}_2, \cdots)]$ and the complement of the section of the given sets is the sum of their complements $[\overline{\mathfrak{D}(A_1, A_2, \cdots)} = \mathfrak{S}(\overline{A}_1, \overline{A}_2, \cdots)]$. Since $P = Q$, $P \subset Q$ (i.e., P is a subset of Q) imply $\overline{P} = \overline{Q}$, $\overline{P} \supset \overline{Q}$, it follows that every equation between sets remains true if every set is replaced by its complement and the symbols $\mathfrak{S}$ and $\mathfrak{D}$ are interchanged; and every inequality remains true after the same changes and the additional interchange of $\supset$ and $\subset$. For example, $A \subseteqq \mathfrak{S}(A, B)$ leads to $\overline{A} \supseteqq \mathfrak{D}(\overline{A}, \overline{B})$ and hence to $A \supseteqq \mathfrak{D}(A, B)$. This simple property Hausdorff calls the "principle of duality"; he utilizes it in various connections to secure results through formulas usually obtained otherwise. We remark here that the author uses formulas to a much larger extent than is customary in the theory of aggregates, one of whose noticeable characteristics is the unusual freedom from calculational methods.

Let $X_1, X_2, \cdots, X_m$ be m given sets. Taking a cue from algebra, we seek a list of simple sets which like the sum $\mathfrak{S}(X_1, X_2, \cdots, X_m)$ and the section $\mathfrak{D}(X_1, X_2, \cdots, X_m)$ involve the given sets symmetrically. Hausdorff calls these sets "symmetrische Grundmengen," and defines them as follows: $A_i (i = 1, 2, \cdots, m)$, the ith such set, consists of the elements that occur in at least i X's. Thus $A_1 = \mathfrak{S}(X_1, X_2, \cdots, X_m)$, $A_m = \mathfrak{D}(X_1, X_2, \cdots, X_m)$. An essential property of these sets is that they are expressible in terms of the X's by means of sums and sections. Their introduction is due to Hausdorff. The author considers various properties of these sets, and in particular, utilizes them in an interesting theory of measure, which was at first planned as final but was later discarded for a more concise treatment; a sketch of the old theory appears in the appendix.

The second chapter (pages 32–45) deals with functions, products, and powers, and their laws of operation.

The third chapter (pages 45–69) treats of the cardinal numbers. Cardinal number is not defined with Cantor as what remains of a set after the individual nature and the order of the elements are abstracted; nor with Russell, as a class of classes. Hausdorff takes the simple and formal point of view, which is clearly the most satisfactory: We associate uniquely with a system of sets A a system of things a—of indifferent nature—in such a way that the same things corre-

spond to two sets if and only if the sets are equivalent. These things or symbols we call cardinal numbers. Two proofs are given of the Bernstein equivalence theorem, the first essentially like that of Bernstein, and the second, according to Zermelo, without the use of the infinite set of integers. The rest of the chapter is devoted to the comparison of cardinal numbers, and includes such theorems as $\aleph^{\aleph_0} = \aleph$, $\aleph_0^{\aleph} = 2^{\aleph} > \aleph$ ($\aleph$ = cardinal number of the continuum), and the theorem of J. König. Famous theorems, such as those of the non-denumerability of the continuum, and of the equivalence of the plane of points and the line of points, frequently appear—as is also the case in other chapters—as very special cases of general considerations, or as side remarks in the current text. For the non-denumerability of the continuum, however, a special proof is added.

Chapter IV (pages 69–101) takes up order; a substantial portion of the ideas and results is due to Hausdorff. After various definitions of simple (= linear) order, the sum of an ordered set (the "argument") of any number of ordered sets, and the product of a finite number of sets are defined, and the laws for operating with these processes given. The subset M of the ordered set A is said to be "coinitial" with A if no element of A exists preceding every element of M. Similarly, "cofinal." M is "dense in A" if for every pair $a < b$ of elements of A there exists a pair of elements $m < n$ of M such that $a \leqq m < n \leqq b$. The decomposition $A = P + Q$, where $P(\neq 0)$ and $Q(\neq 0)$ have no elements in common and every element of P precedes every element of Q, is said to be a "jump" (Sprung) if P has a last element and Q a first; a "gap" (Lücke), if neither P has a last, nor Q a first. A "dense set" (in the "absolute," as contrasted with the "relative" sense) is one without jumps; a "continuous set," one without jumps or gaps. A "scattered set" is one possessing no dense subset. A sum $\Sigma_i{}^J A_i$ $(A_i \neq 0)$ is scattered when and only when the argument J is scattered and each A_i is scattered. Every ordered set is either scattered or the sum of scattered sets over a dense argument. The chapter closes with the discussion of types of order, in particular, of the class of countable dense types, and of continuous types.

The fifth chapter (pages 101–139) is devoted to normally ordered (wohlgeordnete) sets and the ordinal numbers. The treatment comprises comparability of cardinal numbers, trans-

finite induction, powers and products, alephs and the number classes, the initial numbers (Anfangszahlen) and Zermelo's Wohlordnungssatz. The proof of the general theorem $\aleph_a\aleph_a = \aleph_a$ for every ordinal number a, is given in elegant and brief form; the first proof of the theorem, given by Hessenberg in his Grundbegriffe der Mengenlehre, is long and roundabout. It may be remarked that frequently the simplest and most elegant proofs of important theorems are to be found in the Grundzüge, either directly or after appropriate modification of the generalized form in which they usually appear.

For the Wohlordnungssatz both of Zermelo's proofs with unessential but neat modifications are given. Hausdorff has no difficulty—neither has the reviewer—in accepting either of these proofs as rigorous. In fact, as is sometimes the case with the work of mathematicians who have misgivings about the theorem, the multiplicative principle (Prinzip der Auswahl) steals in noiselessly (cf. for example, page 54) before the Wohlordnungssatz is mentioned.

The sixth chapter (pages 139–209) contains a wealth of material mostly from the author's own researches. Unfortunately space will not permit—especially because of the more technical character of the subject matter—a description of these elegant and general results. We must content ourselves merely with mentioning the partially ordered sets, the distinction as related to coinitiality and cofinality of the element and gap characteristics and the consequent classification of ordered sets, the general products and powers of ordered sets and the interesting connection with non-archimedean number systems, as shown by the general theorem of Hahn (*Berichte der Wiener Akademie der Wissenschaften*, 1907).

The remaining chapters of the book (VII–X) will prove of more general interest because they are concerned with the applications of the abstract theory to the study of space relations. It is in these chapters especially that Hausdorff impresses you with his masterful exposition. The theory of point sets is cast into a new and more general mold, and the resulting treatment is characterized throughout by originality, naturalness, and beauty.

Chapter VII (pages 209–260) begins with the statement: "The theory of aggregates has celebrated its most beautiful triumphs in its application to point sets and in the clarification and heightened precision of the fundamental concepts of

geometry; this is admitted even by those who demean themselves skeptically towards the abstract theory of aggregates." The subject matter of the chapter concerns point sets in general spaces. The author's justification of his abstract treatment is as follows (page 210): "Now a theory of spatial point sets would naturally have, in virtue of the numerous accompanying properties, a very special character, and if we wished to confine ourselves from the outset to this single case, we should be obliged to develop one theory for linear point sets, another for planar point sets, still another for spherical point sets, etc. Experience has shown that we may avoid this pleonasm and set up a more general theory comprehending not only the cases just mentioned but also other sets (in particular, Riemann surfaces, spaces of a finite or an infinite number of dimensions, sets of curves, and sets of functions). And, indeed, this gain in generality is associated not with increased complication, but on the contrary, with a considerable simplification, in that we utilize—at least for the leading features—only few and simple assumptions (axioms). Finally we secure ourselves in this logical-deductive way against the errors into which our so-called intuition may lead us; this alleged source of knowledge—the heuristical value of which, of course, no one will impugn—has, as it happens, shown itself so frequently insufficient and unreliable in the more subtle parts of the theory of aggregates, that only after careful examination may we have faith in its apparent testimony."

Hausdorff does not bind himself to a single set of assumptions. The center of interest lies, of course, in the theorems, and the assumptions are graded accordingly, a new assumption or a modification being adopted only when the mathematical content naturally calls for such a change. In the carefully planned march from the abstract in the direction of greater specialization, Hausdorff gives repeated evidence of his mathematical-esthetic insight.

The developments in Chapter VII are based entirely upon the following "neighborhood" postulates (Umgebungsaxiome, page 213). A "neighborhood" is a point set. The abstract set or space E in question is unrestricted except for the postulates:

(A) To every point x of E there corresponds at least one neighborhood U_x; every neighborhood U_x contains the point x.

(B) If U_x, V_x are two neighborhoods of x, there is a neighborhood W_x contained in both.

(C) If the point y lies in U_x, there is a neighborhood U_y lying in U_x.

(D) For $x \neq y$ there are two neighborhoods U_x, U_y with no point in common.

For the euclidean plane, the neighborhoods of a point P may be taken as the circles (exclusive of the boundary) having P as center.

A space satisfying the four neighborhood postulates is called "topological."

There are numerous concrete examples of topological spaces, among which may be mentioned the ordinary euclidean spaces (also after adjunction of the ideal point at infinity), certain spaces in which the distance from point to point is measured on a non-archimedean scale, space of a denumerable infinity of dimensions, and function space. Some of these spaces are also "metric" (see below).

An "inner point" of a set A—belonging to the entire space or "universe" E—is one possessing a neighborhood lying entirely in A. A point of A that is not one of its inner points is a "brink" point (Randpunkt), as distinguished from "boundary" point (Grenzpunkt), which need not belong to the given set. A "region" (Gebiet) is a set every point of which is an inner point of the set; a "brink aggregate" (Randmenge), one every point of which is a brink point. The inner points of the complement B of A ($A + B = E$) are called the "outer" points of A; the boundary points of A consist of the brink points of both A and B. The universe E and every neighborhood is a region. The inner points of an arbitrary set constitute a region; the brink points, a brink aggregate. The sum of any number, and the section of a finite number, of regions are regions.

Connected with the last statement, there is a simple but fruitful principle: If a system of sets M—like the system of regions—has the property that the sum of any number of sets of the system belongs to the system, we may, for any given set A containing at least one M as subset, define the largest M contained in A; if the section of any number of M's is an M, we may for every A contained in at least one M define the smallest M containing A. This principle is used in various connections, for example, in the definition of the "kernel" (see below).

The introduction of the sets A_α, A_β, A_γ—the first has been

little used—leads to considerable formal simplification, and enables Hausdorff to give many short proofs through formal processes instead of direct reflection upon the nature of the hypotheses. x is an α-point of A if every neighborhood U_x contains at least one point of A (which may be x itself); a β-point, if every neighborhood U_x contains an infinite number of points of A; a γ-point, if every neighborhood U_x contains a non-denumerable set of points of A. A_α, A_β, A_γ are the respective totalities of these points (A_β, the derivative of A). That A is "closed" may be expressed by $A \supseteqq A_\beta$, or by $A = A_\alpha$; "dense-in-itself," by $A \subseteqq A_\beta$ or by $A_\alpha = A_\beta$; "perfect," by $A = A_\beta$. The sets A_α, A_β, A_γ are closed. The section of any number of closed sets and the sum of a finite number of closed sets are closed. The sum of any number of sets each dense-in-itself is dense-in-itself. The largest subset of A that is dense-in-itself exists according to the principle just mentioned; it is called the "kernel" of A.

An infinite set without β-points is said to be "divergent"; a set without divergent subsets, "compact." The set A converges to the limit x if every neighborhood of x contains all the points of A with the possible exception of a finite number. A decreasing sequence $A_1 \supseteqq A_2 \supseteqq \cdots$ of compact, closed, non-vanishing sets has a non-vanishing section (Cantor). A compact, closed set contained in the sum of a sequence of regions is contained in the sum of a finite number of these regions (Borel).

A clear and systematic treatment is given of the limits of a sequence of sets $\{A_n\}$. Six different kinds of limits are distinguished: (1) the "lower limit" consists of the points belonging to "nearly all" the A_n, i.e., all with the possible exception of a finite number; (2) the "upper limit," of the points belonging to an infinity of the A_n; (3) the "lower closed limit," of the points (belonging to the A_n or not) every neighborhood of which contains points of nearly every A_n; (4) the "upper closed limit," of the points every neighborhood of which contains points of an infinity of the A_n; (5) the "lower limit region," of the points for which a neighborhood exists belonging (in its entirety) to nearly all the A_n; and (6) "the upper limit region," of the points for which a neighborhood exists belonging to an infinite number of the A_n. Various modes of representation of these limits by means of sums and sections are given.

Another novel feature is the systematic introduction and use of the notion of "relativity." A is said to be (relatively) "closed in M" if it is the section of M and a closed set; a "relative region of M," if it is the section of M and a region. These notions are special cases of a complete theory of relativity, which arises by substituting for the universe E an arbitrary subset M of it. The neighborhoods U_x are replaced by their sections with M, which again satisfy the neighborhood assumptions; M may therefore be regarded as a new universe possessing all the properties of topological spaces. We thus have relative α-points, relative inner points, and so on.

The definition of connectivity differs from those heretofore given, but it is the most desirable in the opinion of the reviewer: A non-vanishing set M is said to be "connected" if it is not expressible as the sum of two sets ($\neq 0$) that are (relatively) closed in M and have no points in common. A "component" of a non-vanishing set is one of its largest connected subsets, i.e., a connected subset contained in no other such subset. The "quasi-component" of A belonging to the point p consists of the points belonging to the same summand as p in every decomposition of A as the sum of two sets closed in A and having the null-set as section. The quasi-components may differ from the components. After a series of theorems on connectivity, the chapter is devoted to density, and to the application of some of the results to sets of real numbers.

In Chapter VIII (pages 260–358) special topological spaces are considered. A stride towards ordinary space is made by the successive introduction of the denumerability postulates:

(E) The set of neighborhoods of x is denumerable for every x.

(F) The totality of all neighborhoods is denumerable.

With the aid of (E), it follows, for example, that every convergent set (= set having a limit) is countable; and that if x is a β-point of a set A, there is a convergent subset of A with x as limit. With (F) the γ-points begin to play an important rôle. If a set has no γ-points belonging to it, it is countable. The following equations hold ($A_{\alpha\alpha}$ = set of α-points of A_α, etc.): $A_{\alpha\alpha} = A_\alpha$, $A_{\alpha\beta} = A_\beta$, $A_{\alpha\gamma} = A_{\beta\gamma}$; $A_{\beta\alpha} = A_\beta$; $A_{\gamma\alpha} = A_\gamma$, $A_{\gamma\beta} = A_\gamma$, $A_{\gamma\gamma} = A_\gamma$. A set of regions no pair of which have common points is countable. There are in all $\aleph$ regions and $\aleph$ closed sets ($\aleph$ = cardinal number of the continuum). The sum of any number of regions is the sum of a countable number (at most) of them.

By means of this result, Borel's theorem may be extended and its complete converse given.

The connection of point set theory with the transfinite ordinals lies in such theorems as the following (proved with the aid of postulate (F)): An ascending normally ordered set $\{S_\xi\}$—i.e., $S_\rho \supset S_\sigma$ for $\rho > \sigma$—of different regions S_ξ is at most countable. Similarly for a descending set of regions and for relative regions, and likewise for closed and relatively closed sets. It is at this juncture that Hausdorff introduces his generalization of "reducible" sets. It turns out that a Hausdorff reducible set is representable as the sum of differences of descending normally ordered closed sets, and conversely.

Further developments refer to metric spaces: "We believe the time has come when a continuation of the neighborhood theory would be accompanied with a loss of simplicity." A metric space satisfies the following postulates, where $\overline{xy}$ denotes the "distance"—Fréchet's écart—from x to y:

(α) (Postulate of symmetry) $\overline{yx} = \overline{xy}$.

(β) (Postulate of coincidence) $\overline{xy} = 0$ when and only when $x = y$.

(γ) (Triangle postulate) $\overline{xy} + \overline{yz} \geqq \overline{xz}$.

Space will not permit a discussion of the rich content of the rest of the chapter. We mention: distances between sets, connectivity properties of metrical spaces including ρ-connectivity, properties related to the various limits of a sequence of sets, Borel sets, conditions for compactness and "complete" (vollständige) spaces. After certain general theorems concerned chiefly with connectivity in euclidean n-space, the chapter turns to the euclidean plane—see quotation in early portion of this review—and after a succession of thirteen carefully graded theorems, culminates in a proof—modeled after that of Brouwer—of the Jordan theorem.

The ninth chapter (pages 358–399) deals with representations or functions. If the original set A and its image B (= totality of elements $f(a)$, where a ranges over A) are topological spaces, the continuity of $f(a)$ is equivalent to the condition that every relative region of B is the image of a relative region of A. If f is continuous and A connected, then B is connected. From this it follows, in particular, that a continuous, real function defined in a connected set—for example, in a linear interval—takes every value between any

two of its values. Among other things, the chapter considers uniform continuity for metrical spaces, continuous curves including the space-filling curves, the character of the points of continuity of a discontinuous function, sequences of functions and the generalizations of theorems of Arzelà and Baire. There are novel results on the classification of functions and the set of convergence points of a sequence of functions.

The tenth and final chapter (pages 399–448) treats of Peano-Jordan content and Borel-Lebesgue measure; both theories are developed in a new, elegant and appreciably similar fashion. There are applications to decimal and to continued fractions. After the deduction of the important properties of Lebesgue integrals, the chapter closes with the proof that a function of limited variation possesses a derivative except in a set of measure zero.

The appendix (pages 449–473) gives references to the literature and contains numerous discussions of substantial content and interest.

Misprints are few in number; and errors, invariably of a minor character. The mention of most of them would be regarded as hypercritical, were it not for the high standards of the author. The careful examination of the reviewer has brought to light only the following: page 11, line 15, "Elemente" instead of "Punkte"; page 28, line 27, "Relationen" instead of "Gleichungen"; page 58, line 14 and page 105, line 20, "Denn" instead of "Dann"; page 272, G_{i_1} is not printed clearly; page 366, line 6, insert "Kap. VIII" after "nach"; page 442, line 5, insert Σ before $\mu_i\delta_i$. On pages 85 and 291, in the footnotes, occur the equations $m = \{m\}$, $x = \{x\}$; the author clearly intimates their objectionability, but it would be better to bar altogether such illogical statements. On page 106, the author speaks of numbers greater than W, when he means numbers greater than numbers of W. On page 229, he forgets to discuss A_k at the end of the section. On page 276, the restriction that λ shall be a limiting number does not enter the proof. This is an exception to prove the rule that in the Grundzüge every word counts. On page 444, the statement that X is normally ordered, although later cleared up, contains at first an ambiguity; for X may be normally ordered without being ordered according to magnitude. On page 470, $\psi^{-m_1}\delta\psi^{m_1}$ is not necessarily an α, but may be a γ; it is an easy matter, however, to fill out the gap.

As for more important criticism, one may quarrel with the author for his abstract style, for his euclidean manner of grading the proofs, so that no difficulties remain and none but mild climaxes are reached, for his finish that may excite admiration but hardly activity on the reader's part. One may crave for a book that is built like a drama around a single idea —a more sketchy book, leaving more to the reader's imagination, a book with a less diversified and more emphatic message. But such remonstrance would be like quarrelling with Beethoven for having written symphonies instead of operas. There is no such thing as *the* book. Hausdorff's Grundzüge is a treatise, and as a treatise it necessarily falls short of the summum bonum. But as a treatise it is of the first rank.

HENRY BLUMBERG.

SHORTER NOTICES.

The Casting-Counter and the Counting Board. By FRANCIS PIERREPONT BARNARD. Oxford, Clarendon Press, 1916. 358 pp. + 63 plates.

When we consider that Gerbert, the greatest mathematician living in Europe at the close of the tenth century, wrote upon the use of counters as an aid to computation; that Robert Recorde, who is often called the founder of the English school of mathematicians of the sixteenth century, did the same; and that nearly all computation in Europe before the year 1500 (in Italy before c. 1200) was performed by the aid of some type of abacus, we may well infer that the "casting-counter," as Professor Barnard calls it, has played an important rôle in the history of calculation. Indeed, our very word "calculate" is, it need hardly be said, due to this very fact, the word "calculus" meaning a pebble, calculi being used in numerical work in the classical period of the Greek and Roman civilizations.

When we also consider the fact that it was the bamboo rods, used by the early Chinese algebraists to express coefficients, that suggested to the Japanese the *sangi* which were used for the same purpose, and also suggested the idea of determinants which their scholars developed in the seventeenth

century, anticipating the expansion of these forms by Leibniz, the counter takes on a dignity that might hardly have been anticipated. And finally when we consider that it was the humble abacus that led to such devices as the suan pan, the soroban, the tschotü, and the choreb now used in, numerically speaking, the greater part of the civilized world; that led Pascal to invent and Leibniz to work upon the modern calculating machine; and that finds frequent place in the literature of Chaucer, Shakespeare, and other makers of our language, it may well occur to the mathematician to look with interest upon the treatise under review, which Professor Barnard has published in such sumptuous form.

The counters that have come down to us date from the thirteenth to the eighteenth century, the earlier pieces having been melted up for the metal or lost because of the very fact that they were too common to be held in much esteem. There are, of course, many disks of bone, baked earth, or metal that have come down to us from ancient times and which may have been used for purposes of calculation by the calculones, the calculatores, or the numerarii; but since they bear no inscriptions, we are uncertain of their use and they may have been merely pieces employed in playing such ancient games as backgammon or checkers. From the thirteenth century on for five hundred years, however, specimens were preserved, and these still remain to tell the story, decade by decade, of the use of these devices, of their change in form, of the Rechenmeisters for whom they were struck, and of the slow decay of the abacus as the power of the eastern numerals came to be the better understood. To-day, about all that we have in common use to remind us of the ancient and medieval counter (projectilis, jetton, augrim stone, calculus, abaculus, Worpghelt, Legpennig, Rechenpfennig, méreau à compte, tessera ad computandum, and the like) of our ancestors—is the poker chip, the wire with its disks above the billiard table, and the string of beads used in various religious ceremonials throughout the Buddhist, Mohammedan, and Christian world.

In the work under review Professor Barnard has given us the benefit not only of knowing about his collection of some 7,000 jettons and about some 40,000 other specimens which he has examined, but of benefitting by his rare scholarship in all that relates to medieval history. As to the former, he

has reproduced by photographic process a hundred and twenty-six of the rarest counters in his collection, generally both in obverse and reverse, and has given scientific descriptions of a large number of others. As to the opportunity that he has given the reader of having some share in his scholarship—he has shown the painstaking care in research into historical details, and the interesting style of presenting facts that characterize his other writings upon the period to which he has devoted a life of study. The fact that the bibliography of works consulted contains more than six hundred items is proof of the care with which he has considered the literature relating to the subject.

The work consists of three parts. Part I relates to the history of the casting-counter and to descriptions of the specimens found in England, Italy, France, the Low Countries, Germany, and Portugal. Part II contains the only worthy description that we have of the counting boards and counting cloths used in the medieval and Renaissance periods, and is admirably illustrated by numerous plates. Part III sets forth the methods of casting with jettons, a subject upon which we have plenty of information from such sources as Robert Recorde (c. 1542), John Awdeley (1574), Nicholas Cusa (1514), Martinus Siliceus (1526), Köbel (1514), and various other writers of the sixteenth century, upon whose works the author has freely drawn.

There are two elaborate indexes, one of legends and inscriptions on the counters and the other of a general nature—and nothing is more conducive to the comfort of a student who has occasion to consult a work of this nature than a good index.

It may be said of the work as a whole that it represents the most elaborate study that we have upon any of the minor features of this kind in the history of mathematics, and that it deserves a place in every college and university library and on the shelves of every one who is working in the special field of the history of computation. DAVID EUGENE SMITH.

Tables des Nombres Premiers, et de la Décomposition des Nombres de 1 *à* 100,000. By G. INGHIRAMI, reviewed and corrected by Dr. PROMPT. Gauthier-Villars et Cie., 1919. xi + 35 pp.

THIS little factor table gives the smallest divisor of all

numbers less than 100,000 which are prime to 2 and 5. The inclusion of the smallest factor 3 adds little to the value of the table and much to the bulk. It is decidedly inferior to Burckhardt's table in convenience of arrangement, and is approximately twice as bulky as a table which omits multiples of 2, 3, 5 and 7. It is difficult to see what end is served by the republication of such a table at the present time.

The book also contains a table of "Tessaréen" numbers, which seem to be prime numbers of the form $a^2 - a - 1$. The connection between such numbers and the numbers representable by the binary quadratic form $x^2 - xy - y^2$ is not indicated. A list of such numbers with the corresponding values of a is given, the list extending as far as the prime 19991. What particular use is to be made of this table is not indicated. The "preliminary explanation" avoids giving any demonstration of the properties of "Tessaréen" numbers, on the ground that such a demonstration would not teach anything to one who was already familiar with the theory of numbers, and would not be understood by one who was not.

The author of the preliminary explanation—name not signed, but perhaps Lebon—deplores the neglect of the theory of numbers, which neglect he attributes to the vicious methods and barbarous terminology of German mathematicians. Gauss especially comes in for a thorough castigation for his bizarre and incoherent formulas!

Five pages of the introduction are devoted to a biography of Inghirami by Giovanni Giovannozzi. Inghirami's most important work seems to have been done in astronomy and in geodesy. The factor tables here republished appeared for the first time in 1832 at the end of a volume on Elementi di Matematiche. He evidently did not know of Burckhardt's tables published some twenty years previously.

D. N. LEHMER.

Lectures of the Theory of Plane Curves. By SURENDRAMOHAN GANGULI, M.Sc., Lecturer in Pure Mathematics, University of Calcutta. Part I, x + 140 pp.; Part II, xiii + 350 pp., and 13 pages of figures. Published by the University of Calcutta, 1919.

THESE lectures were delivered to postgraduate students and comprise in a fairly satisfactory manner most of the topics usually presented in an elementary course on plane curves.

The first part is concerned with the general theory, the second with cubics and quartics. In teaching the subject, Ganguli had constant recourse to the classic treatises of Salmon and Clebsch, and the works of Basset, Scott, and others, and laid particular stress on Sylvester's theory of residuation. But instead of using this theory for the systematic study of the geometry of point groups on curves, Ganguli merely makes occasional applications of it. The definition of a curve based upon the representation of a function, in §§ 10–11, is confusing and inadequate. In the first place there is no proper postulation of the space in which curves are to be defined. Metric and projective concepts are indiscriminately mixed up, so that, as a consequence, a proper formulation of the systems of coordinates is impossible. From the present prevailing critical point of view this is one of the weakest points of Ganguli's lectures. What, for example, do the following statements mean, when the graph of a function is not defined? "In the modern theory of functions, it is held that a function can be completely defined by means of a graph arbitrarily drawn in the finite and continuous domain of the independent variable." Again, notice the erroneous declaration, "The modern theory of functions says that the equation $F(x, y) = 0$ cannot in general represent a curve, it can do so if y can be expressed as a regular (or rational) function $f(x)$ of x, i.e., if $f(x)$ is a continuous, finite and differentiable and separately monotonous function. It is only by a combination of these conditions that $y = f(x)$ can represent a curve." From this it is evident that the author still relies on the hazy notions about a curve as they were held at the time of Euler. Not the slightest attention is given to complex domains, or to what kind of an equation $F(x, y) = 0$ is.

The principle of duality and the corresponding use of line coordinates are brought in incidentally and without an effort at systematic representation. Using the designation "reciprocal curves" suggests the idea of the special duality involved in polar reciprocity. Moreover the lack of a proper projective space makes an adequate treatment of the behavior of curves at infinity impossible.

The typography of both volumes could be considerably improved in an eventual future edition. Much more attention should also be given to the reading of proofs.

But aside from the defects mentioned above the beginner

may learn a good deal about the properties of algebraic curves, so that in this respect the publication of a new English treatise on curves is not without value, and deserves commendation.

ARNOLD EMCH.

Nouvelles Méthodes de Résolution des Equations du 3e Degré. By le VTE. DE GALEMBERT. Paris, Vuibert, 1919. 22 pp.

This pamphlet gives a method for rapid numerical calculation of the real roots of the cubic equation

$$x^3 + Ax^2 + Bx + C = 0,$$

new as far as I am aware, for the case in which there are three real roots. The equation is reduced to the form $z^3 - 3z = y$; the latter equation has three real roots if $|y| \leqq 2$. A table of corresponding values of z and y is computed once for all, by means of which the values of z may be found accurately to two decimals, whenever y is known to six places. An additional table gives z and y in terms of x, A, B and C, for the different combinations of signs which the coefficients may have in the general equation. The actual calculation of the roots is very much simplified in this way.

A. DRESDEN.

The Integral Calculus. By JAMES BALLANTYNE. Boston, Published by the author, 1919. 41 pp.

THE subtitle of this small book is sufficiently descriptive of its scope. It reads, "On the integration of the powers of transcendental functions, new methods and theorems, calculation of Bernouillian numbers, rectification of the logarithmic curve, integration of logarithmic binomials, etc."

The author has several new series expansions of transcendental functions; but does not burden his tale with arguments as to the rigor of his methods or the general validity of his formal results. The book gives the impression of having been written for the fun of it, by a very ingenious gentleman, who was having a fine time giving free rein to his analytical processes and going gladly wherever those steeds—dangerous if unchecked—might lead him.

The style of the text may be indicated by such expressions of olden time flavor as, "integrals of even powers of sin xdx to radius 1"; "the value of C is the area of the full quadrant

of the curvilinear"; "the two curves $y = 1/\sin^m x$ and $y = 1/\cos^m x$ having their origin at opposite ends of the axis of $x \cdots$"; "when m is an odd negative, there will always appear in the series one irrational term, namely $\pm \frac{1}{0} \tan^0 x$"; "that is, the logarithm of an infinite number, multiplied by zero = nothing."

In section 1 various reduction formulas for integrals of $\sin^m x$, $\cos^m x$, etc., are developed and also formulas expressing those integrals in series of trigonometric functions, e.g.: "The integral of $\sin^m xdx$ for any value of m is

$$A(1 - \cos x) - \frac{B}{3}(1 - \cos^3 x) + \frac{C}{5}(1 - \cos^5 x) - \frac{D}{7}(1 - \cos^7 x) + \cdots,$$

where $A, B, C, D, \cdots$ are the successive terms in the development of the binomial $(1 + 1)^{(m-1)/2}$." These finite and series developments are equated in section 2 to derive some well known and new formulas. For example, if we let $x = \frac{1}{2}\pi$ in the above series, for $m = 0$, and equate it to $\int \sin^0 xdx = x$, we obtain

$$1 + \frac{1}{2\cdot 3} + \frac{3}{2\cdot 4\cdot 5} + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \cdots = \pi/2.$$

Again by equating $\int \cos^0 xdx = x$ to

$$\int \cos^m xdx = A \sin x - \frac{B}{3}\sin^3 x + \frac{C}{5}\sin^5 x \cdots,$$

we have

$$x = \sin x + \frac{1}{2\cdot 3}\sin^3 x + \frac{3}{2\cdot 4\cdot 5}\sin^5 x \cdots$$

"an inversion of the series given by Newton for the sine in terms of the arc."

From the series expansion, for m odd,

$$\int_0^{\pi/2} \sin^m xdx = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdots\frac{m-1}{m}$$

and from the finite expansion, for m even,

$$\int_0^{\pi/2} \sin^m xdx = \frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\frac{m-1}{m}\right)$$

the author concludes, "But if m be considered infinite, the distinction between odd and even values of m vanishes and these two expressions will be equal to each other." It follows that

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9}\cdots,$$

the expression given by Wallis.

Section 3 takes up the series for expressing the tangent in terms of the arc and its relation to the Bernoullian numbers. From $\int \tan x dx = -\log\cos x$, the author gets

$$\tan x = x + \frac{2}{2\cdot 3}x^3 + \frac{16}{2\cdot 3\cdot 4\cdot 5}x^5 + \frac{272}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \cdots$$

and works out a scheme for finding the numerators N_a of the terms of this expansion, namely,

$$N_a = \pm 1 \mp aN_1 \pm \frac{a(a-1)(a-2)}{2\cdot 3}N_3 \mp \frac{a(a-1)(a-2)(a-3)(a-4)}{2\cdot 3\cdot 4\cdot 5}N_5 \pm \cdots.$$

to the term N_{a-2}, where a is the index of the power of x whose coefficient is required and using the upper or lower sign according as $\frac{1}{2}(n-1)$ is even or odd. The author concludes this section with a practical method for the calculation of the Bernoullian numbers. The derivation of the formula is by the theory of differences, but it gives the numbers with more facility than the usual formulas. This formula is

$$1 = \frac{1}{P+1} + \frac{1}{2} + \frac{P}{2}B_1 - \frac{P(P-1)(P-2)}{2\cdot 3\cdot 4}B_2 + \frac{P(P-1)(P-2)(P-3)(P-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}B_3 - \cdots,$$

a series terminating at $B_{P/2}$. By assuming positive, even values for P the numbers B_i may be read off.

In section 4 integrals of powers of transcendental functions in terms of these functions are given. The procedure is as follows: Let $y = \varphi(x)$. Express dy/dx in terms of y and

integrate $y^m/(dy/dx)$ with respect to y (usually in series form) "as an independent variable and as an algebraic function." The result will be the integral of $\varphi^m(x)dx$ expressed in terms of the function itself, whether it be algebraic or transcendental. The formulas derived at this point are of immediate applicability to the quadrature and rectification of, for example, such troublesome curves as the logarithm.

The final section on "the integration of the logarithm of binomials and other complex quantities" is less satisfactory and of less interest than the earlier sections and is made up of various odds and ends.

E. GORDON BILL.

NOTES.

AT the meeting of the National Academy of Sciences held at Princeton University November 16–17, 1920, the following mathematical papers were read: By E. B. WILSON: "Equipartition of energy"; by EDWARD KASNER: "Einstein gravitational fields: orbits and light rays"; by J. W. ALEXANDER: "Knots and Riemann spaces"; by PHILIP FRANKLIN: "The map coloring problem."

THE July number (volume 42, number 3) of the *American Journal of Mathematics* contains the following papers: "The failure of the Clifford chain," by W. B. CARVER; "On the representations of numbers as sums of 3, 5, 7, 9, 11 and 13 squares," by E. T. BELL; "On a certain class of rational ruled surfaces," by ARNOLD EMCH.

THE opening (September) number of volume 22 of the *Annals of Mathematics* contains: "On multiform functions defined by differential equations of the first order," by PIERRE BOUTROUX; "Hermitian metrics," by J. L. COOLIDGE; "On the expansion of certain analytic functions in series," by R. D. CARMICHAEL; "Notes on the cyclic quadrilateral," by F. V. MORLEY: "Note on the preceding paper," by F. MORLEY; "Qualitative properties of the ballistic trajectory," by T. H. GRONWALL.

It is announced that because of the greatly increased cost of printing the publication of the *Journal de Mathématiques pures et appliquées*, founded by Liouville in 1836, will be discontinued with the current volume (series 8, volume 1) unless a considerable number of new subscriptions are secured at once. If any American mathematicians or libraries have suspended their subscriptions because of war conditions, they should send in their orders for renewal as soon as possible.

The following six American doctorates in mathematics should be added to the list published in the November Bulletin, making a total of twenty-one conferred in 1919–20: H. R. Brahana, Princeton: "Curves on surfaces"; H. J. Ettlinger, Harvard: "I. Existence theorems for the general real self-adjoint linear system of the second order. II. Oscillation theorems for the real self-adjoint linear system of the second order"; E. S. Hammond, Princeton: "Periodic conjugate nets of curves"; G. M. Robison, Cornell: "Divergent double sequences and series"; Bird M. Turner, Bryn Mawr: "Plane cubics with a given quadrangle of inflexions"; W. L. G. Williams, Chicago: "Fundamental systems of formal modular seminvariants of the binary cubic."

Professor J. H. Tanner, of Cornell University, and Mrs. Tanner have given to the trustees of that institution fifty thousand dollars to establish a mathematical institute under the following stipulations. The money is to be allowed to accumulate without diversion for seventy-five years. At the end of that time one professor shall be appointed, whose duty it shall be to begin the formulation of plans for the proposed institute. At the end of each of the four succeeding periods of five years one or more additional professorships shall be established, the incumbents to collaborate in the same plans. The stipends of these professors shall be paid from the fund, but no other demands shall be made upon it until one hundred years from the date of the deed of gift (June, 1920), from which time on the income of the entire sum shall be devoted to the maintenance of the institute; half of the expenditure of each year is to be applied to research in the mathematical sciences.

It is announced that a new edition of Professor H. S. Carslaw's book on Fourier's series and integrals, published

in 1906 by Macmillan and for some time out of print, is now in press. The book has been completely rewritten and will consist of two volumes, the first dealing with infinite series and integrals, with special reference to Fourier's series and integrals, and the second with the conduction of heat. It is expected that the first volume will appear early in 1921, and the second in the course of that year.

APPLICATIONS for appointment to the two Benjamin Peirce instructorships at Harvard University for the year 1921–22 should reach Professor W. F. OSGOOD on or before February 1, 1921, accompanied by the necessary papers. Each instructor is now required to give two elementary courses and one other course ordinarily of an advanced character, but has no administrative duties. The salary has been increased to $2000.

THE Paris academy of sciences has awarded its Poncelet prize to Professor ELIE CARTAN, of the University of Paris, and its Francœur prize to Professor RENÉ BAIRE, of the University of Dijon.

THE Italian society of sciences (the XL) has awarded its gold medal for 1920 to Professor A. SIGNORINI, of the University of Palermo, for his papers published during the last five years.

THE new Greek mathematical society, founded in 1918, now numbers about one hundred members.

PROFESSOR ALBERT EINSTEIN, of the University of Berlin, has accepted the chair of science at the University of Leyden. He will divide his time between the two institutions.

PROFESSOR M. DISTELI has been appointed professor of geometry at the University of Zurich.

PROFESSOR F. GONSETH has been appointed professor of mathematics at the University of Bern.

PROFESSOR MICHEL PLANCHEREL, of the University of Freibourg, has been appointed to a professorship at the Zurich technical school, as successor to the late Professor A. HURWITZ.

At the University of Cagliari, Dr. A. Comessatti, of the University of Padua, and Dr. M. Picone, of the University of Catania, have been appointed to associate professorships of mathematics.

Dr. A. Palatini, of the University of Padua, has been appointed associate professor of rational mechanics at the University of Messina.

Professor G. Armellini, of the University of Padua, has been transferred to a full professorship of celestial mechanics at the University of Pisa.

Ppofessor C. J. de la Vallée Poussin has been elected president of the recently formed Fédération belge des sociétés des sciences mathématiques, physiques, naturelles, médicales et appliquées.

Professor Charles Riquier, of the University of Caen, has been elected correspondent of the Paris academy of sciences in the section of geometry, as successor to the late Professor H. G. Zeuthen, and Professor L. T. Quevede, of the University of Madrid, has been elected correspondent in the section of mechanics, as successor to the late Professor Boulvin.

In the faculty of sciences of the University of Paris, the following changes have been made. The professorship of the theory of functions has been changed to a professorship of theoretical and celestial physics. Dr. Emile Borel, professor of the theory of functions, has been appointed professor of the calculus of probabilities and mathematical physics, as successor to Professor V. J. Boussinesq, who has retired. Dr. Paul Painlevé, professor of rational mechanics, has been appointed professor of analytical and celestial mechanics, as successor to Professor Paul Appell; Professor Elie Cartan succeeds Professor Painlevé in the chair of rational mechanics, and Professor Ernest Vessiot, recently appointed assistant director of the Ecole normale supérieure, succeeds Professor Cartan in the chair of the differential calculus. Dr. J. Drach has been appointed professor of general mathematics, and Dr. Paul Montel maître de conférences in mathematics.

PROFESSOR PIERRE BOUTROUX has been appointed professor of the history of sciences at the Collège de France.

PROFESSOR GABRIEL KOENIGS has been elected member of the Conseil supérieure de l'instruction publique, as successor to Professor APPELL.

PROFESSOR M. G. HUMBERT, of the Ecole polytechnique, has retired from active teaching, with the title of honorary professor.

PROFESSOR E. FABRY, of the University of Montpellier, has been appointed professor of integral and differential calculus at the University of Aix-Marseille.

DR. RENÉ GARNIER, of the University of Poitiers, has been promoted to a professorship of theoretical and applied mechanics.

MR. A. E. JOLIFFE, of Corpus Christi College, Oxford, has been appointed professor of mathematics at the Royal Holloway College, London.

LT. COL. A. R. RICHARDSON has been appointed professor of mathematics at University College, Swansea.

PROFESSOR L. E. DICKSON was appointed delegate of the National academy of sciences to attend the conference called by the Royal society of London in September to consider the future of the International catalogue of scientific literature.

AT Colgate University, associate professor A. W. SMITH has been made full professor and head of the department of mathematics as successor to Professor J. M. TAYLOR, who has retired. Professor T. R. AUDE, of the Carnegie Institute of Technology, has been appointed associate professor of mathematics.

PROFESSOR J. K. LAMOND, of Pennsylvania College, Gettysburg, has resigned to accept a position in the engineering department of the Bell Telephone Company at Philadelphia.

At the U. S. Naval Academy, assistant professor G. R. Clements has been promoted to an associate professorship of mathematics, and Dr. L. S. Dederick and Dr. L. T. Wilson to assistant professorships.

At the University of Wyoming, Professor C. B. Ridgaway has retired, after twenty-four years of service, and Professor C. E. Stromquist succeeds him as head of the department of mathematics. Mr. A. R. Fehn, of the Kansas Agricultural College, has been appointed associate professor of mathematics.

Associate professor H. H. Conwell, of the University of Idaho, has resigned to accept a similar position in the department of mathematics at Beloit College.

Dr. Lennie P. Copeland and Dr. Mary F. Curtis have been promoted to assistant professorships of mathematics at Wellesley College.

At the University of North Carolina, Professor William Cain has retired from active teaching, and Professor Archibald Henderson succeeds him as head of the department of mathematics; assistant professors A. W. Hobbs and J. W. Lasley, Jr., have been promoted to associate professorships.

In the department of mathematics at Purdue University, the following changes have been made: assistant professor W. A. Zehring has been promoted to an associate professorship; Mr. C. T. Hazard, Mr. C. K. Robbins, and Dr. G. H. Graves have been promoted to assistant professorships; Dr. E. M. Berry, Mr. W. H. Frazier, Mr. W. R. Hardman, and Mr. J. A. Needy have been appointed instructors.

At the University of Saskatchewan, Dr. I. A. Barnett, of the University of Illinois, has been appointed assistant professor of mathematics. Dean G. H. Ling has been granted leave of absence for the current year, and is at present in England.

Professor I. L. Miller, of Carthage College, has been appointed associate professor of mathematics at the South Dakota State College.

Assistant professor Hillel Halperin, of the University of Arkansas, has been appointed to an associate professorship at the Texas Agricultural and Mechanical College.

Professor J. B. Faught, of the State normal college at Kent, Ohio, has been appointed professor of mathematics at Yankton College.

Professor P. H. Graham, of Agnes Scott College, has been appointed instructor in mathematics at New York University.

Mr. R. M. Mathews, of the University of Minnesota, has been appointed assistant professor of mathematics at Wesleyan University.

Mr. B. D. Roberts has been appointed professor of mathematics at Parsons College, Fairfield, Iowa.

Mr. C. D. Ehrman, of the University of Wisconsin, has been appointed assistant professor of mathematics at Richmond College.

Miss Frances B. Hatcher has been appointed associate professor of mathematics at Westhampton College, Richmond, Va.

Dr. H. F. MacNeish has been promoted to an assistant professorship of mathematics at the College of the City of New York.

Dr. C. L. E. Wolfe, of the Junior College at Santa Rosa, has been appointed instructor in mathematics at the California Institute of Technology, Pasadena.

Professor H. W. Stager, of Fresno Junior College, has been appointed instructor in mathematics at the University of Washington.

At Harvard University the following instructors in mathematics have been appointed for the current academic year: professor C. A. Garabedian, of New Hampshire College; Mr. R. E. Langer, Mr. E. L. Mackie, and Mr. H. Levy, of

Harvard University; Mr. A. J. Cook, of the University of Alberta.

Mrs. M. I. Logsdon has been appointed associate in mathematics at the University of Chicago.

Miss Mary Ball has been appointed instructor in mathematics at Northwestern University.

Dr. Tobias Dantzig, of Johns Hopkins University, has resigned to undertake mathematical research for an engineering company in New York City.

Mr. T. H. Johnson, Mr. A. S. Adams, and Mr. W. L. Lucas have been appointed instructors in mathematics at the University of Maine.

Mr. J. B. Rosenbach, of the University of Illinois, has been appointed instructor in mathematics at the Carnegie Institute of Technology.

Miss Jessie M. Short, formerly of Carleton College, has been appointed instructor in mathematics at Reed College.

Professor John Perry, of the Royal College of Science, London, died August 4, 1920, at the age of seventy years.

Professor F. A. Tarleton, of Trinity College, Dublin, died June 20, 1920, at the age of seventy-nine years.

The death is reported, in August, 1920, of assistant professor H. D. Frary, of the University of Wisconsin.

Professor Samuel Hanaway, who retired in 1916 on account of ill health from the department of mathematics of the College of the City of New York, died recently at the age of sixty-six years.

Professor E. W. Stanton, of the teaching staff of Iowa State College since 1870, died September 12, 1920, at the age of seventy years.

Professor Maelynette Aldrich, of Martha Washington College, died February 22, 1920.

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THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION.

THE thirty-sixth regular meeting of the San Francisco Section was held at the University of California on Saturday, October 23. The chairman of the Section, Professor Blichfeldt, presided at the early part of the meeting; Professor Lehmer presided at the latter part. The attendance was twenty-two, including the following fifteen members of the Society:

Professor R. E. Allardice, Professor B. A. Bernstein, Professor H. F. Blichfeldt, Professor Florian Cajori, Professor M. W. Haskell, Professor Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. F. R. Morris, Professor C. A. Noble, Professor T. M. Putnam, Dr. Pauline Sperry, Dr. S. E. Urner, Dr. A. R. Williams.

The following officers were elected for the year: chairman, Professor D. N. Lehmer; secretary, Professor B. A. Bernstein; programme committee, Professors H. F. Blichfeldt, W. A. Manning, B. A. Bernstein.

The dates of the next two meetings were provisionally fixed as April 9, 1921, and October 22, 1921.

The following papers were presented:

(1) Professor D. N. LEHMER: "On inverse ternary continued fractions."

(2) Professor M. W. HASKELL: "Curves autopolar with respect to two conics."

(3) Professor FLORIAN CAJORI: "Historical note on notations for ratio and proportion."

(4) Professor E. T. BELL: "The elliptic modular equation of the third order and the form $x^2 + 3y^2$."

(5) Professor E. T. BELL: "The reversion of class number relations and the total representation of an integer as a sum of square or triangular numbers."

(6) Professor E. T. BELL: "Class numbers and the form $xy + yz + zx$."

(7) Professor E. T. BELL: "Singly infinite class number relations."

(8) Professor E. T. BELL: "On recurrences for sums of divisors."

(9) Mr. H. W. BRINKMANN: "The group characteristics of the ternary linear fractional group and of various other groups."

(10) Professor H. F. BLICHFELDT: "Notes on geometry of numbers."

Mr. Brinkmann was introduced by Professor Manning. In the absence of the author the papers of Professor Bell were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. A ternary continued fraction is defined by means of its partial quotient pairs $(p_1, q_1;\ p_2, q_2;\ p_3, q_3;\ \cdots;\ p_n, q_n)$, the successive convergent sets (A_n, B_n, C_n) being solutions of the difference equation $u_n = q_n u_{n-1} + p_n u_{n-2} + u_{n-3}$, where the initial values for A_n, B_n, C_n are respectively, 1, 0, 0; 0, 1, 0; 0, 0, 1. (See the *Proceedings of the National Academy of Sciences*, volume 4, pages 360–364, December, 1918.)

It is of importance in the theory of cubic irrationalities to know what relations must exist among the partial quotient pairs in order that the two inverse ternary continued fractions $(p_1, q_1;\ p_2, q_2;\ \cdots;\ p_n, q_n)$, $(p_n, q_n;\ p_{n-1}, q_{n-1};\ \cdots;\ p_1, q_1)$ should have the same characteristic cubic, the characteristic cubic being defined by the equation

$$\begin{vmatrix} A_{n-2} - \rho & B_{n-2} & C_{n-2} \\ A_{n-1} & B_{n-1} - \rho & C_{n-1} \\ A_n & B_n & C_n - \rho \end{vmatrix} = 0.$$

Professor Lehmer finds, besides the obvious case where the pairs read the same backward and forward, that the cubic is the same for both fractions if $p_i = \alpha t + \beta$, $q_i = \gamma t + \delta$, where α, β, γ, δ are any fixed positive or negative integers or zero, while t is a variable parameter.

2. Professor Haskell described the configuration of four mutually autopolar conics, and gave a method for finding an unlimited number of higher plane curves autopolar with respect to two conics, when the latter are autopolar with respect to each other.

3. Professor Cajori notes that the colon was first used for ratio by the astronomer Vincent Wing in 1651. The forerunner

of Oughtred's notation $A \cdot B :: C \cdot D$ to signify $A : B = C : D$ is found in Billingsley's Euclid, 1570, where we find $9 \cdot 6 : 12 \cdot 8$. With Billingsley, the dot and colon were symbols of punctuation which were not yet limited to specific arithmetical use. Isaac Barrow wrote $A \cdot B + C \cdot D$ to signify the "compounding of ratios," i.e., $A : B$ times $C : D$. John Wallis opposed this practice. Sometimes, in the same equation, Barrow used $+$ to indicate the multiplication of ratios as well as the addition of terms.

4. In Professor Bell's first paper it is shown that the modular equation for the transformation of the third order in elliptic functions is implied by a theorem relating to even functions of two variables, the arguments of the functions being linear functions of the integers which represent an arbitrary integer in the form $x^2 + 3y^2$. The theorem admits of easy extension (for the appropriate quadratic forms), to the modular equation of the nth order, and takes particularly simple forms when n is a prime $4k + 1$, $12k + 7$ or the triple of a prime $12k + 1$. The paper will be published in the *Bulletin* of the Greek Mathematical Society.

5. In Professor Bell's second paper it is shown that the class number relations of Kronecker, Hermite and others may be reversed so as to give the class number for a negative determinant explicitly in terms of the total number of representations of certain integers each as a sum of square or triangular numbers, and further it is shown that each of a pair of such expressions (a relation and its inversion) implies the other. Recurrences for the computation of the new functions relating to total numbers of representations are given. These bear a striking resemblance to those for the class numbers.

6. In Professor Bell's third paper three general class number relations, each of which admits of specialization in an infinity of ways, are derived and their connection with the arithmetical form $xy + yz + zx$ considered. Each relation involves class numbers and an arbitrary even function of a single variable. On particularizing the function in various ways several special class number theorems are found at once, the simplest of which is: the total number of representations of n in the form

$xy + yz + zx$ for which $x, y, z > 0$, is $3[G(n) - 1]$ when and only when n is prime, $G(n)$ being the whole number of classes of binary quadratic forms for the determinant $-n$. The next simplest cases are famous class number relations due either to Kronecker or Hermite; the next are of the Liouville types, and thence onwards the special consequences are class number relations of kinds not hitherto stated. The paper will appear shortly in the *Tôhoku Mathematical Journal*.

7. A set of seventeen closely interrelated class number formulas of the kind described in Professor Bell's third paper is derived. In several ways the set is complete, no more results of the same general sort being implicit in the analysis. On specializing the arbitrary even functions involved in the most obvious ways, all of Kronecker's and Hermite's formulas drop out as the simplest cases, also some of those due to Liouville and Humbert. The method used was explained in the first part of a paper presented to the Society in December, 1918, which will appear in the *Transactions*.

8. A considerable section of Chapter X (volume I) of Dickson's History of the Theory of Numbers is devoted to divisor recurrences obtained from the expansions of elliptic functions. Professor Bell shows in this paper that any such recurrence is a very special case of a general relation between the divisors concerned, and specific application is made (in one of the illustrations) to well-known formulas of Halphen and Glaisher. A curious lacuna is observed: none of the formulas is sufficient for the calculation by recurrence alone of the $2r$th ($r = 0, 1, 2, \cdots$) powers of *all* the divisors, although the like may easily be done for the $(2r - 1)$th powers.

9. Mr. Brinkmann determines the group characteristics of the group of all ternary linear fractional substitutions of determinant unity whose coefficients are marks of any Galois field, and also the group characteristics of the group of all ternary linear fractional substitutions of non-vanishing determinant. Further, the group characteristics of all primitive permutation groups of degree less than or equal to fifteen are determined, so far as they are not known already.

10. In Minkowski's development of geometry of numbers

(cf. this BULLETIN, volume 25 (1919), page 449) the following theorem is fundamental: if we designate by a Minkowski surface in R_n a finite surface in space of n dimensions, having as its chief characteristic a center of symmetry toward which it is nowhere convex (cf. l. c. for specific definition), then a Minkowski surface in R_n and of volume $\geqq 2^n$ will contain at least three distinct lattice points (i. e., points whose coordinates are integers) if its center is a lattice point. In order to extend the usefulness of the geometry of numbers, Professor Blichfeldt has amplified this theorem to read as follows: (1) a Minkowski surface in R_n of volume $\geqq 2^n k$ and whose center is a lattice point, must contain more than $k - 1$ distinct pairs of lattice points in addition; (2) a Minkowski surface in R_n which contains k lattice points, its center being one, must have a volume $> (k - n)/n!$, if these k points do not all lie on a linear R_{n-1}. Some applications of this theorem were presented.

B. A. BERNSTEIN,
Secretary of the Section.

AN IMAGE IN FOUR-DIMENSIONAL LATTICE SPACE OF THE THEORY OF THE ELLIPTIC THETA FUNCTIONS.

BY PROFESSOR E. T. BELL.

(Read before the San Francisco Section of the American Mathematical Society June 18, 1920.)

1. In his memoir on "Rotations in space of four dimensions"* Professor Cole defined a system of four mutually orthogonal lineoids yzw, xzw, xyw, xyz (which we shall denote by X, Y, Z, W respectively) through a point O, the four lines and six planes determined by these, and with reference to this system found the transformations into itself of a sphere S with center at O. Henceforth we assume the radius of S to be $\sqrt{n}$, where n is an integer > 0. From this system we shall derive an image of the theory of the elliptic theta func-

* *Amer. Jour. of Math.*, vol. 12 (1890), p. 191.

tions by considering the reflexions of certain point configurations C, C', C'', C''', C^{iv} lying upon S with respect to O and the bisectors of the angles between X, Y, Z, W, a bisector of an angle between two lineoids being defined as a locus of points equidistant from the two. The space about O is latticed by four systems of lineoids parallel respectively to X, Y, Z, W, the successive lineoids in each system being at unit distances apart. We shall call this the unit lattice L. Any point all of whose coordinates are integers belongs to L; and conversely L contains only such points. Any integer > 0 being in several ways a sum of four integral squares, S always contains points of L, and these are symmetrical in pairs with respect to O.

Denote by L' the lattice containing all those points and only those whose coordinates are $(4a, 4b, 4c, 4d)$, where a, b, c, d take all integral values (including zero) from $-\infty$ to $+\infty$, and by $\alpha\beta\gamma\delta$ the lattice derived from L' by successive translations of L' through distances α, β, γ, δ parallel respectively to X, Y, Z, W, where α, β, γ, δ are integers $\geqq 0$, so that L' is 0000. There clearly are in all precisely 256 distinct lattices $\alpha\beta\gamma\delta$, each of which is contained in L, and these may be represented by symbols $\alpha'\beta'\gamma'\delta'$, where α', β', γ', δ' are the positive residues mod 4 of α, β, γ, δ. For brevity we assign current numbers to a special set of 64 contained in the 256. Only the $\alpha'\beta'\gamma'\delta'$ wherein α', β' are both even or both odd are required in the sequel, and likewise for γ', δ'. The requisite half-symbols $\alpha'\beta'$, $\gamma'\delta'$ are therefore 00, 02, 11, 13, 20, 22, 31, 33. Write

$$20, 22, 02, 00 \equiv 2, 4, 6, 8,$$

$$11, 13, 31, 33 \equiv 1, 3, 5, 7,$$

respectively. In this notation the lattice 0222 is 64; the current number of 3320 is 72; that of 1311 is 31; 3100 is 58; 0231 is denoted by 65, etc. To signify that all the points of a certain configuration C belong to one of these lattices, say to ij, we give C the corresponding double suffix, C_{ij}. The theory of the theta functions is formally equivalent to the symmetries of certain C_{ij} lying upon S. By the formal equivalence of A, B we mean that each implies the other.

2. Consider first any point P lying within (not on any of X, Y, Z, W) any one T of the sixteen right tetraedral angles into which space is partitioned by X, Y, Z, W. Bisect the

angles which any one of X, Y, Z, W, say X, makes with the remaining three, and denote by Y', Z', W' those parts of the bisectors which lie within T, and by U', V', U'', V'', U''', V''' the like parts of the bisectors of the angles between the three pairs of opposite pairs X, Y and Z, W; X, Z and Y, W; X, W and Y, Z, viz., U' is the bisector for X, Y; V' for Y, Z, etc. (We have chosen the internal bisectors with respect to the angles of T.) Reflect P in Y', reflect the image in Z', and reflect this image in W', getting finally the point P_1. In whatever order the three successive reflexions are performed, it is clear that the same P_1 is reached. From P_1 in the same way derive P_2, from P_2 similarly P_3, and from P_3 in the same way P_4. Then $P_4 \equiv P$. Second, if P lies on at least one of X, Y, Z, W, we avoid ambiguities (of sign) by requiring the reflexions to be performed so that the signs of the coordinates of P are unchanged; e.g., the signs being $(++--)$ are to be the same before and after reflexion. Reflect P in O, getting P_0; join the centroid Π of P_0, P_1, P_2, P_3 to O, and produce $O\Pi$ through Π to cut S in P', which point we shall call the mate of P. Reflect P' in U', reflect the image in V', getting P'', called the first skew mate of P. Similarly from P' and U'', V'' get P''', and from U''' and V''', get P^{iv}, the second and third skew mates of P. Note that $P', \cdots, P^{\text{iv}}$ are significant only with respect to the particular T in which P lies. Taking the mates of all points in any configuration C we get its mate C', and similarly for the first, second and third skew mates C'', C''', C^{iv}.

We shall be concerned with two kinds of symmetry about O of C_{ij} and their mates. If A, B are any configurations such that each may be brought into coincidence with the other by reflexions in O of some (or all) of its points, A, B are called images of each other. All those points of any configuration C which are such that no one of them is the reflexion in O of another, form a configuration called the residue of C; and any configurations A_1, B_1 are said to be skew images of each other when their residues coincide.

Finally we need also the idea of lattice configurations with multiple points. Let each point of $C_1, C_2, \cdots, C_r$ belong to the unit lattice. By $C_1 + C_2 + \cdots + C_r, \equiv C$, we mean the configuration which consists of all the points of $C_1, C_2, \cdots, C_r$. A point occurring in precisely s of the C_i is multiple of order s in C; and two coincident configurations are identical when

and only when points occupying the same position in both are of equal multiplicities. In particular if A, B are images or skew images of each other, the points in any pair which are regarded as reflexions in O of one another must be of the same multiplicity. To indicate that each point in C is of multiplicity s, we write sC.

We can now give the image of the single theta functions. The geometrical theorems will first be stated, their theta equivalents then pointed out, and the means for passing from one to the other briefly indicated. The geometry can be derived simply from first principles. It is shorter, however, to proceed as in § 4. The eleven images can be compressed into one relating to S and the unit lattice, but the statement is complicated. The eleven exhibit a manifold symmetry recalling that of crystals, which becomes evident when the C_{ij} and their mates and skew mates C_{ij}', C_{ij}'', etc., are written with the suffixes in full, thus, C_{0011}, C_{3320}'. In all that follows the C_{ij}, and therefore also their mates and skew mates, are configurations of points lying on the S of radius $\sqrt{n}$ defined in § 1.

3. The first two theorems concern the case $n \equiv 0 \bmod 4$, and the configurations

$$C_0 \equiv C_{22} + C_{26} + C_{62} + C_{66}, \quad C_2 \equiv C_{24} + C_{28} + C_{64} + C_{68},$$

$$C_4 \equiv C_{44} + C_{48} + C_{84} + C_{88}, \quad C_6 \equiv C_{42} + C_{46} + C_{82} + C_{86};$$

$$C_1 \equiv C_{11} + C_{17} + C_{71} + C_{77}, \quad C_3 \equiv C_{13} + C_{15} + C_{73} + C_{75},$$

$$C_5 \equiv C_{33} + C_{35} + C_{55} + C_{53}, \quad C_7 \equiv C_{31} + C_{37} + C_{51} + C_{57}.$$

THEOREM I. *Each of $C_0 + C_4$, $C_1 + C_5$ is the image of its mate, and each of $C_2 + C_6$, $C_3 + C_7$ is the image of the mate of the other.*

THEOREM II. *The configurations in each of the following pairs are images of each other:*

$$C_0 + C_2 + C_4' + C_6' + C_1 + C_3 + C_5' + C_7',$$

$$C_0' + C_2' + C_4 + C_6 + C_1' + C_3' + C_5 + C_7;$$

$$C_0 + C_2 + C_2' + C_4' + C_1' + C_5 + C_7 + C_7',$$

$$C_0' + C_4 + C_6 + C_6' + C_1 + C_3 + C_3' + C_5'.$$

The next two are for $n \equiv 2 \bmod 4$, and the configurations

$$B_1 \equiv C_{12} + C_{16} + C_{72} + C_{76}, \qquad B_3 \equiv C_{34} + C_{38} + C_{54} + C_{58},$$

$$B_5 \equiv C_{14} + C_{18} + C_{74} + C_{78}, \qquad B_7 \equiv C_{32} + C_{36} + C_{52} + C_{56}.$$

THEOREM III. *$B_1 + B_3$ is the image of its second skew mate, and $B_5 + B_7$ is the image of its mate.*

THEOREM IV. *$B_1 + B_3'''$ is the image of $B_1''' + B_3$, and $B_5' + B_7$ is the image of $B_5 + B_7'$.*

The next is also for $n \equiv 2 \bmod 4$, and the configurations

$$D_1 \equiv C_{16} + C_{18} + C_{36} + C_{38} + C_{52} + C_{54} + C_{72} + C_{74},$$

$$D_2 \equiv C_{12} + C_{14} + C_{32} + C_{34} + C_{56} + C_{58} + C_{76} + C_{78}.$$

THEOREM V. *The following are skew images of each other:*

$$2D_1 + D_1' + D_2'' + D_2''' + D_2^{\text{iv}},$$

$$2D_2 + D_2' + D_1'' + D_1''' + D_1^{\text{iv}}.$$

4. The foregoing theorems imply the theory of the theta functions. For, if f, g are single-valued functions of four variables existing when each variable takes integral values $\gtreqless 0$, such that

$$f(x, y, z, w) = \quad f(-x, -y, -z, -w),$$

$$g(x, y, z, w) = -g(-x, -y, -z, -w),$$

and otherwise are wholly arbitrary, we may express that A, B are images, that A, B are skew images, by

$$\Sigma f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \Sigma f(\beta_1, \beta_2, \beta_3, \beta_4),$$

$$\Sigma g(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \Sigma g(\beta_1, \beta_2, \beta_3, \beta_4)$$

respectively, the Σ's extending to all points $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of A, and $(\beta_1, \beta_2, \beta_3, \beta_4)$ of B. On remarking that if $P \equiv (x_1, x_2, x_3, x_4)$, then $P', P'', P''', P^{\text{iv}} = (x_1', x_2', x_3', x_4')$, (x_2', x_1', x_4', x_3'), (x_3', x_4', x_1', x_2'), (x_4', x_3', x_2', x_1'), where $x_i' \equiv s - x_i$ and $2s = x_1 + x_2 + x_3 + x_4$, we may easily verify that the five theorems are equivalent to the following analytical restatements of them. The m_i denote odd integers $\gtrless 0$, the l_i even integers $\gtreqless 0$, representing $n > 0$ in the forms

$$\sum_{i=1}^{4} m_i^2, \qquad \sum_{i=1}^{4} l_i^2, \qquad m_1^2 + m_2^2 + l_3^2 + l_4^2,$$

and summations are with respect to all l_i, m_i for a given n. The rest of the notation is:

$$2\lambda_1 = l_1 + l_2,\quad 2\lambda_3 = l_3 + l_4,\quad 2\mu_1 = m_1 + m_2,\ 2\mu_3 = m_3 + m_4,$$

$$\lambda\ = \lambda_1 + \lambda_3,\quad \mu\ = \mu_1 + \mu_3,\quad \nu\ = \mu_1 + \lambda_3,\quad 2\alpha = m_1 - 1 + l_3,$$

$$l_i' = \lambda - l_i,\quad m_i' = \mu - m_i,\quad l_i'' = \nu - l_i,\qquad m_i'' = \nu - m_i.$$

Corresponding to theorems (I)–(V) we now have the following:

$$\text{(I}'\text{)}\quad \Sigma[(-1)^{\mu} + 1]f(m_1, m_2, m_3, m_4) = \Sigma[(-1)^{\mu} + 1]f(m_1', m_2', m_3', m_4'),$$

$$\Sigma[(-1)^{\mu} - 1]f(m_1, m_2, m_3, m_4) = \Sigma[(-1)^{\lambda} - 1]f(\ l_1',\ l_2',\ l_3',\ l_4'),$$

$$\Sigma[(-1)^{\lambda} + 1]f(\ l_1,\ l_2,\ l_3,\ l_4) = \Sigma[(-1)^{\lambda} + 1]f(\ l_1',\ l_2',\ l_3',\ l_4'),$$

$$\Sigma[(-1)^{\lambda} - 1]f(\ l_1,\ l_2,\ l_3,\ l_4) = \Sigma[(-1)^{\mu} - 1]f(m_1', m_2', m_3', m_4').$$

$$\text{(II}'\text{)}\quad \Sigma[(-1)^{\mu_1}f(m_1, m_2, m_3, m_4) + (-1)^{\lambda_1}f(l_1, l_2, l_3, l_4)] = \Sigma[(-1)^{\mu_1}f(m_1', m_2', m_3', m_4') + (-1)^{\lambda_1}f(l_1', l_2', l_3', l_4')],$$

$$\Sigma[(-1)^{\mu_1}f(m_1, m_2, m_3, m_4) - (-1)^{\lambda_1}f(l_1, l_2, l_3, l_4)] = \Sigma[(-1)^{\mu_3}f(m_1', m_2', m_3', m_4') - (-1)^{\lambda_3}f(l_1', l_2', l_3', l_4')].$$

$$\text{(III}'\text{)}\quad \Sigma[(-1)^{\nu} + 1]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^{\nu} + 1]f(l_3'', l_4'', m_1'', m_2''),$$

$$\Sigma[(-1)^{\nu} - 1]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^{\nu} - 1]f(m_1'', m_2'', l_3'', l_4'').$$

$$\text{(IV}'\text{)}\quad \Sigma[(-\ )^{\mu_1} + (-1)^{\lambda_3}]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^{\mu_1} + (-1)^{\lambda_3}]f(l_3'', l_4'', m_1'', m_2''),$$

$$\Sigma[(-1)^{\mu_1} - (-1)^{\lambda_3}]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^{\mu_1} - (-1)^{\lambda_3}]f(m_1'', m_2'', l_3'', l_4'').$$

$$\text{(V}'\text{)}\quad 2\Sigma(-1)^{\alpha}g(m_1, m_2, l_3, l_4) = \Sigma(-1)^{\alpha}[g(l_4', l_3', m_2', m_1') + g(l_3', l_4', m_1', m_2') + g(m_2', m_1', l_4', l_3') - g(m_1', m_2', l_3', l_4')].$$

To pass to the theta functions, replace $f(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $g(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ by the cos, sin respectively of $(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4)$, where x_1, x_2, x_3, x_4 are parameters. In this form the trigonometric sums involving unaccented letters in I′–IV′, and the left of the identity in V′, are readily seen to be the coefficients of q^{4n} or q^{4n+2} in products of the form

$$\vartheta_\alpha(x_1, q^4)\vartheta_\beta(x_2, q^4)\vartheta_\gamma(x_3, q^4)\vartheta_\delta(x_4, q^4),$$

where $\alpha, \beta, \gamma, \delta$ are certain of the numbers 0, 1, 2, 3. Again, the sums involving accented letters in the trigonometric forms of I′–IV′ and the right of V′ are likewise seen to be the coefficients of the same powers of q collected from four such theta products, taken with appropriate signs, in which the variables are $x_i' \equiv s - x_i$, where $2s = x_1 + x_2 + x_3 + x_4$. In this way we find I′–V′ in their trigonometric cases to imply eleven theta identities, which, as they are easily accessible in H. J. S. Smith's second paper on the multiplication formula of four theta functions (Papers, volume 2, page 279), we need not transcribe. It will be sufficient to state the particular formulas of Smith which the trigonometric forms of I′–V′ thus imply. The results in I′ give Smith's (i) $\pm$ (ii),(iii) $\pm$ (iv), and therefore (i) $-$ (iv) in his set A. Similarly his B, C are implied by our II′, III′; his (ix), (xi) of D by our IV′, and his (x) by our V′. From these eleven independent theta formulas, the theory of the theta and elliptic functions, as is well known, follows readily; in fact Smith so derives the theory in his "Memoir on the theta and omega functions" (Papers, volume 2, page 415). The analogous derivation by Jacobi in a famous memoir (Werke, volume 1, page 499) differs only in details. His set (A) is the equivalent of Smith's A–D. We have followed Smith's development rather than Jacobi's because it is the more symmetrical. Jacobi's gives another geometrical image of the theory, and Kronecker's well known exposition of Jacobi's methods yet a third, in which the simpler regular solids inscribed in S play an interesting part.

Now conversely I′–V′, and therefore I–V are implied by the eleven theta formulas of Smith. This follows immediately from the method of paraphrase* outlined in this BULLETIN, volume 26, page 220, § 13. Hence I–V imply, and are implied by, the theory of the elliptic theta functions; viz., the two are formally equivalent.

* The proofs of the method are given in "Arithmetical paraphrases," Part I, to appear shortly in the *Transactions*

5. There is an analogous image for the theory of the theta functions of r variables. In it the lattice space is of $2r(r+1)$ dimensions, and the appropriate configurations lie on a system of r four-dimensional spheres immersed in the higher space. For $r > 1$ the image is not expressible in terms of reflexions alone.

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NOTE ON THE MEDIAN OF A SET OF NUMBERS.

BY PROFESSOR DUNHAM JACKSON.

(Read before the American Mathematical Society September 7, 1920.)

LET $a_1, a_2, \cdots, a_n$ be a set of real numbers, which may or may not be all distinct. Let

$$S_2(x) = \sum_{i=1}^{n} (x - a_i)^2.$$

The value of x which reduces $S_2(x)$ to a minimum is the arithmetical mean of the numbers $a_1, \cdots, a_n$. If the condition that $S_2(x)$ be a minimum is replaced by the condition that

$$S_1(x) = \sum_{i=1}^{n} |x - a_i|$$

be reduced to a minimum, the median of the a's is obtained. It is uniquely defined whenever n is odd; if the numbers a_i are arranged in order of magnitude, so that

$$a_1 \leqq a_2 \leqq \cdots \leqq a_n,$$

and if $n = 2k - 1$, the median is simply a_k, the middle one of the a's. The median is uniquely defined also when n is even, $n = 2k$, if it happens that $a_k = a_{k+1}$, being then equal to this common value. Otherwise, the definition is satisfied by any number x belonging to the interval

$$a_k \leqq x \leqq a_{k+1},$$

and the median is to this extent indeterminate.

The purpose of the following paragraphs is to show that

for each value of $p > 1$ there is a definite number $x = x_p$ which minimizes the sum

$$S_p(x) = \sum_{i=1}^{n} |x - a_i|^p,$$

and that x_p approaches a definite limit X as p approaches 1. The value of X coincides with the median as already defined, in the cases where that definition is determinate; and when $n = 2k$ and $a_k \neq a_{k+1}$, X is a definite number between a_k and a_{k+1}. It serves thus to supplement the former definition, on the theoretical side at any rate.

It is not unlikely that the same average has been discussed, as a curiosity at least, by writers on statistics. The accurate proof of the statements involved, however, seems to be an exercise in pure mathematics. It will have to be admitted that the number does not lend itself readily to direct computation, except in the simplest cases.

In the first place, for any particular value of $p > 1$, $S_p(x)$ is a continuous function of x which is always positive or zero, and which becomes infinite as x goes out to infinity in either direction. Consequently it attains a minimum for some one value of x at least.

Furthermore, each term $|x - a_i|^p$ has a continuous derivative for all values of x, the value $x = a_i$ not excepted. Hence $S_p(x)$ also has a continuous derivative everywhere, and this derivative must vanish at any point where a minimum is reached. But the derivative of $|x - a_i|^p$ always increases when x increases; hence the same thing is true of the derivative of $S_p(x)$, and $S_p'(x)$ can vanish only once. That is, the minimizing value x_p is indeed uniquely determined, and is moreover characterized by the vanishing of $S_p'(x)$.

To be more precise with regard to the formulas involved,

$$\frac{d}{dx}|x - a_i|^p = \frac{d}{dx}(x - a_i)^p = \quad p(x - a_i)^{p-1}, \qquad x > a_i;$$

$$\frac{d}{dx}|x - a_i|^p = \frac{d}{dx}(a_i - x)^p = -\, p(a_i - x)^{p-1}, \qquad x < a_i;$$

while for $x = a_i$ the derivative is correctly represented by either formula, being equal to zero. If i is any one of the indices $1, 2, \cdots, n - 1$, and x any number in the interval

$a_i \leqq x \leqq a_{i+1},$

$$S_p(x) = (x - a_1)^p + \cdots + (x - a_i)^p + (a_{i+1} - x)^p + \cdots + (a_n - x)^p,$$

$$(1) \quad \frac{1}{p} S_p'(x) = (x - a_1)^{p-1} + \cdots + (x - a_i)^{p-1} - (a_{i+1} - x)^{p-1} - \cdots - (a_n - x)^{p-1}.$$

Now suppose n is odd, $n = 2k - 1$. Let ϵ be any positive quantity, and let r be an index such that

$$a_r < a_k + \epsilon \leqq a_{r+1}.$$

Then $r \geqq k$. By the formula just written down,

$$\frac{1}{p} S_p'(a_k + \epsilon) = (a_k + \epsilon - a_1)^{p-1} + \cdots + (a_k + \epsilon - a_r)^{p-1} - (a_{r+1} - a_k - \epsilon)^{p-1} - \cdots - (a_n - a_k - \epsilon)^{p-1}.$$

When p approaches 1, each of the first k terms on the right approaches 1 as a limit. Any one of the remaining $k - 1$ terms, sign included, has the limit 1, 0, or -1, as the case may be. At any rate,

$$\lim_{p=1} S_p'(a_k + \epsilon) > 0.$$

There exists a positive δ_1 such that

$$S_p'(a_k + \epsilon) > 0$$

whenever $1 < p \leqq 1 + \delta_1$.

Similarly, there is a positive δ_2 such that

$$S_p'(a_k - \epsilon) < 0$$

if $1 < p \leqq 1 + \delta_2$. If δ is the smaller of δ_1, δ_2, and $1 < p \leqq 1 + \delta$, it is certain that $S_p'(x)$ vanishes between $a_k - \epsilon$ and $a_k + \epsilon$. But the point where $S_p'(x)$ vanishes is the point x_p; consequently

$$a_k - \epsilon < x_p < a_k + \epsilon$$

for $1 < p \leqq 1 + \delta$, that is,

$$\lim_{p=1} x_p = a_k.$$

Analogous reasoning shows that when $n = 2k$ and $a_k = a_{k+1}$,

$$\lim_{p=1} x_p = a_k = a_{k+1}.$$

It remains to consider the case of principal interest, the case that $n = 2k$ and $a_k < a_{k+1}$. By reasoning similar to that already employed, it is seen that $\lim_{p=1} S_p'(a_k) < 0$. For at least k terms in the expression for $S_p'(a_k)$ approach the limit -1, the term $(x - a_k)^{p-1}$ is zero, and not more than $k - 1$ terms are positive. In the same way, $\lim_{p=1} S_p'(a_{k+1}) > 0$, and therefore

$$a_k < x_p < a_{k+1}$$

when p is sufficiently near to 1.

When $x > a_i$, $(x - a_i)^{p-1}$ can be represented by the series

$$\begin{aligned}(x - a_i)^{p-1} &= e^{(p-1)\log(x-a_i)} \\ &= 1 + (p - 1) \log (x - a_i) + \tfrac{1}{2}(p - 1)^2 \log^2 (x - a_i) + \cdots \\ &= 1 + (p - 1) \log (x - a_i) + (p - 1)^2 \varphi_i(x, p),\end{aligned}$$

where $\varphi_i(x, p)$ is a function which remains finite if x is fixed and p approaches 1. Similarly,

$$(a_i - x)^{p-1} = 1 + (p - 1) \log (a_i - x) + (p - 1)^2 \varphi_i(x, p)$$

when $x < a_i$. The work so far has been based on the use of the first term as an approximation to the value of the series; it will be convenient now to make use of the explicit form of the first two terms.

If the expressions just indicated are substituted in the right-hand member of (1), for a value of x between a_k and a_{k+1}, there will be k terms each equal to $+1$ and k terms each equal to -1, which will cancel, and a relation will be obtained which can be written in the form

$$\begin{aligned}\frac{1}{p(p-1)} S_p'(x) &= \log (x - a_1) + \cdots + \log (x - a_k) \\ &\qquad - \log (a_{k+1} - x) - \cdots - \log (a_n - x) \\ &\qquad + (p - 1)\varphi(x, p) \\ &= \log \frac{(x - a_1) \cdots (x - a_k)}{(a_{k+1} - x) \cdots (a_n - x)} + (p-1)\varphi(x, p),\end{aligned}$$

where $\varphi(x, p)$ is a function which remains finite for fixed x as p approaches 1.

The fraction on the right in the last relation is equal to 0 for $x = a_k$; it increases steadily when x increases from a_k

toward a_{k+1}, each factor of the numerator increasing while each factor of the denominator decreases; and it becomes infinite as x approaches a_{k+1}. Hence there is just one intermediate point, $x = X$, at which the fraction is equal to 1 and its logarithm to 0.

For $x = X - \epsilon$, where ϵ is an arbitrarily small positive quantity, the logarithm has a negative value, independent of p. Hence $S_p'(X - \epsilon)$ is negative if $p - 1$ is sufficiently small. Similarly, $S_p'(X + \epsilon)$ is positive for small values of $p - 1$. This means that x_p is between $X - \epsilon$ and $X + \epsilon$ if p is sufficiently near to 1, that is,

$$\lim_{p=1} x_p = X.$$

So the original assertion is proved.

The number X is characterized by the equation

$$(X - a_1) \cdots (X - a_k) = (a_{k+1} - X) \cdots (a_n - X).$$

If $n = 2$,

$$X = \frac{a_1 + a_2}{2}.$$

If $n = 4$,

$$X = \frac{a_4 a_3 - a_2 a_1}{(a_4 + a_3) - (a_2 + a_1)}.$$

When $n > 4$, the explicit determination of X will involve the solution of an algebraic equation of more or less high degree.

The discussion has been concerned primarily with values of p not much greater than 1. The proof of the existence and uniqueness of x_p, however, applies equally well when p is arbitrarily large. It is readily seen that

$$\lim_{p=\infty} x_p = Y = \tfrac{1}{2}(a_1 + a_n),$$

the intermediate a's having no effect on the result. For if ϵ is fixed and p is very large, the single term $(x - a_1)^p$ will be greater than the whole sum $S_p(Y)$ when $x \geqq Y + \epsilon$, with a corresponding inequality on the other side, so that x_p must be between $Y - \epsilon$ and $Y + \epsilon$.

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NOTE ON CLOSURE OF ORTHOGONAL SETS.

BY PROFESSOR O. D. KELLOGG.

(Read before the American Mathematical Society April 24, 1920.)

1. *Introduction.*

THE present note has to do with the set $[\varphi_i(x)]$ $(i = 0, 1, 2, \cdots)$ of solutions of a differential equation

$$(1) \qquad \frac{d}{dx}(k\varphi') + (\lambda g - l)\varphi = 0,$$

in which k, g, and l are continuous, and $k > 0$ $(a \leqq x \leqq b)$, $g > 0$ $(a < x < b)$, the solutions satisfying a pair of homogeneous linear self-adjoint boundary conditions

$$(2) \qquad \begin{aligned} U_1(\varphi) &= a_1\varphi(a) + a_2\varphi'(a) - a_3\varphi(b) - a_4\varphi'(b) = 0, \\ U_2(\varphi) &= b_1\varphi(a) + b_2\varphi'(a) - b_3\varphi(b) - b_4\varphi'(b) = 0, \end{aligned}$$

the condition for self-adjointness being

$$(3) \qquad k(a)(a_3b_4 - a_4b_3) = k(b)(a_1b_2 - a_2b_1).$$

Such a set is called *closed* with respect to functions of a given class, provided there is no function of the class orthogonal to all the functions of the set, i.e., no function f such that $\int_a^b f\varphi_i g\,dx = 0$, $(i = 0, 1, 2, \cdots)$. Closure of a set is evidently implied whenever it is known that a function of a given class is uniquely determined by its generalized Fourier constants, and in this way a large number of closure theorems are at hand. Stekloff* has proved that for the case of the Sturm-Liouville boundary conditions, in which $a_3 = a_4 = b_1 = b_2 = 0$, the set of solutions is closed with respect to the class of functions that are integrable with integrable squares. Hilbert,† using boundary conditions in part special cases of (2) above, and in part going beyond, in that singularities of the differential

* *Annales de la Faculté des Sciences de Toulouse*, 2d ser., vol. 3 (1901); *Mémoires de l'Académie Impériale des Sciences de St.-Petersbourg*, vol. 30, No. 4 (1911).

† *Göttinger Nachrichten*, 1904, p. 222.

equation at the end points of the interval (a, b) are considered, shows that the sets studied are closed with respect to the class of continuous functions. In the following, I wish first to call attention to an easily derived identity (7) from which it may be inferred immediately that *if the set of solutions of the differential system* (1) *and* (2) *is closed with respect to the class C of continuous functions not zero, it is then also closed with respect to the class S of summable* functions not null functions.* A second paragraph will indicate an application of the method of successive approximations from which the existence of the set of solutions may be inferred, as well as its closure with respect to functions of class S.

2. *The Relative Closure Identity.*

Let $f(x)$ be a function of class S, and let μ be a value of λ for which the system (1) and (2) has no solution except 0. Then the differential equation

$$\frac{d}{dx}(kw') + (\mu g - l)w = fg \tag{4}$$

has a solution in the following sense: there exists a function $w(x)$, continuous, together with its first derivative, which satisfies the boundary conditions (2), and which satisfies (4) except at most at points of a set of measure 0. To see this, take two solutions, $v_1(x)$ and $v_2(x)$, of the homogeneous equation obtained from (4) by replacing the right-hand member by 0; these may be chosen so that $k(x)(v_1v_2' - v_1'v_2) \equiv 1$. Then

$$w_0(x) = \int_a^x \begin{vmatrix} v_1(\xi), & v_2(\xi) \\ v_1(x), & v_2(x) \end{vmatrix} f(\xi)g(\xi)d\xi \tag{5}$$

is continuous, since the indefinite integral of a summable function is continuous. The same is true of its derivative. Finally, kw', since the indefinite integral of a summable function has a derivative equal to the integrand, except at most at points of a set of zero measure (Lebesgue, l. c., page 124–125), has a derivative except at points of a null set. It is immediately seen that in this sense $w_0(x)$ is a particular solution of (4).

* In the sense of Lebesgue. See Leçons sur l'Intégration et la Recherche des Fonctions primitives, p. 115.

From $w_0(x)$ one forms the solution which satisfies (2):

$$(6)\quad w(x) = \begin{vmatrix} w_0(x) & v_1(x) & v_2(x) \\ U_1(w_0) & U_1(v_1) & U_1(v_2) \\ U_2(w_0) & U_2(v_1) & U_2(v_2) \end{vmatrix} \div \begin{vmatrix} U_1(v_1) & U_1(v_2) \\ U_2(v_1) & U_2(v_2) \end{vmatrix},$$

where the denominator determinant does not vanish, since μ was a parameter value for which the homogeneous problem was not possible. Let now $\varphi_i(x)$ be any one of the set of solutions of (1) and (2), and let λ_i be the corresponding value of λ. From (1) and (4) by a familiar process, we get

$$\frac{d}{dx} k(\varphi_i w' - \varphi_i' w) + (\mu - \lambda_i) w \varphi_i g = f \varphi_i g.$$

This equation, failing at most on a set of measure 0, becomes an identity upon integration between the limits a and b. Since φ_i and w satisfy (2), which are self-adjoint, and (3), the integrated terms disappear, and we are led to

$$(7)\qquad (\mu - \lambda_i) \int_a^b w \varphi_i g dx = \int_a^b f \varphi_i g dx,$$

the identity referred to in the introduction. Since f is not identically zero, w cannot be, by (4), and so is of class C. The identity shows that w is orthogonal to every function φ_i to which f is orthogonal and the theorem stated follows.

3. *Existence and Closure of the Set* $[\varphi_i(x)]$.

We start with the functions f and w above, without, however, supposing anything about their orthogonality to functions of the set $[\varphi_i(x)]$. We define a sequence $w_1, w_2, w_3, \cdots$ as follows:

$$w_{i0} = \int_a^x \begin{vmatrix} v_1(\xi), & v_2(\xi) \\ v_1(x), & v_2(x) \end{vmatrix} \bar{w}_{i-1}(\xi) g(\xi) d\xi,$$

$$(8)\quad w_i = \begin{vmatrix} w_{i0}(x) & v_1(x) & v_2(x) \\ U_1(w_{i0}) & U_1(v_1) & U_1(v_2) \\ U_2(w_{i0}) & U_2(v_1) & U_2(v_2) \end{vmatrix} \div \begin{vmatrix} U_1(v_1) & U_1(v_2) \\ U_2(v_1) & U_2(v_2) \end{vmatrix},$$

$$\bar{w}_i = w_i \Big/ \sqrt{\int_a^b w_i^2 g dx},$$

$$w_1 = w.$$

These functions satisfy at all points the differential equations

$$\frac{d}{dx}(kw_i') + (\mu g - l)w_i = \overline{w}_{i-1}g. \tag{9}$$

We proceed to consider the existence and properties of limit functions of the sequence $[w_i]$. We first note that w_i and w_i' are bounded by a number B independent of i. For, if M is greater than the maxima of $|v_1|$, $|v_1'|$, $|v_2|$, $|v_2'|$, and g, we have by Schwarz's inequality applied to (8_1), $|w_i|$ and $|w_i'| < 2M\sqrt{M(b-a)}$, so that a bound, B, for $|w_i|$ and $|w_i'|$ is easily obtainable.

Next, writing the equation (9) and the same equation with $i+1$ replacing i, multiplying these equations respectively by $\overline{w}_{i+1}$ and $\overline{w}_i$, subtracting, and integrating from a to b, we find $\int_a^b w_{i+1}\overline{w}_{i-1}gdx = \int_a^b w_i\overline{w}_igdx$. If in this equation we replace the value of the normed functions by their values in terms of the un-normed functions, we find

$$\int_a^b w_{i+1}w_{i-1}gdx = \sqrt{\int_a^b w_i^2gdx \cdot \int_a^b w_{i-1}^2gdx}. \tag{10}$$

From this we infer two things. First, by Schwarz's inequality, that $\int_a^b w_i^2gdx \leqq \int_a^b w_{i+1}^2gdx$, and hence, that the bounded numbers $\int_a^b w_i^2gdx$ approach a limit c^2, and secondly, that the integral $\int_a^b w_iw_{i+2}gdx$ also approaches c^2.

We now know that because of the boundedness of the derivatives of w_i, there exists a sub-sequence taken from the w_{2i} which approaches a limit W_0.* This limit is approached uniformly, and is continuous. If we denote by $[w_{2i'}]$ the subsequence approaching W_0 uniformly, then (8) shows that $w_{2i'+1}$ also approaches a limit W_1, and that this function satisfies the differential equation

$$\frac{d}{dx}(kW_1') + (\mu g - l)W_1 = W_0g/c. \tag{11}$$

* Cf. Osgood, *Annals of Mathematics*, vol. 14 (1913), p. 182.

Similarly $w_{2i'+2}$ approaches a limit, W_2, say. But this limit cannot differ from W_0, since

$$\int_a^b (w_{2i'} - w_{2i'+2})^2 g dx = \int_a^b w_{2i'}^2 g dx - 2\int_a^b w_{2i'} w_{2i'+2} g dx + \int_a^b w_{2i'+2}^2 g dx$$

approaches zero, as we have just seen. Hence

$$\int_a^b (W_2 - W_0)^2 g dx = 0,$$

and the continuous functions W_2 and W_0 are equal. We infer further that

$$\frac{d}{dx}(kW_0') + (\mu g - l)W_0 = W_1 g/c. \tag{12}$$

The norms of W_0 and W_1 are equal to c, so that either their sum or their difference is not identically zero. But $W_0 + W_1$ is a solution of (1) and (2) corresponding to $\lambda = \mu - 1/c$ and $W_0 - W_1$ is a solution corresponding to $\lambda = \mu + 1/c$.

Having one solution of the homogeneous problem, we need merely start with a function f orthogonal to it in order to arrive by the above process at a new one. In this way may be inferred the existence of an infinite set of solutions. A simple procedure would be to start with a function $f(x)$ having a break in its derivative but continuous. As such a function cannot be a finite sum of functions $c_i\varphi_i(x)$, it may, after $\varphi_1, \varphi_2, \varphi_3, \cdots$ have been found, be made successively orthogonal to these by subtraction of proper multiples of them, without reducing to zero, and so yield, in time, any function of the set.

The set containing all the solutions of (1) and (2) is then closed with respect to the class S. For, if not, there would be a function f of class S orthogonal to all φ_i, and starting the process developed above with this f, we should ultimately arrive at the contradiction that $W_0 + W_1$ or $W_0 - W_1$, that is, one of the φ_i itself, was orthogonal to the set.

CAMBRIDGE, MASS.,
August 23, 1920.

THE MATHEMATICAL WORK OF THOMAS JAN STIELTJES.

Œuvres complètes de Thomas Jan Stieltjes. Publiées par les soins de la Société Mathématique d'Amsterdam. P. Noordhoff, Groningen. Tome I, 1914: viii + 471 pp. Tome II, 1918: iv + 604 pp.

ONE can not assert that Thomas Jan Stieltjes was one of the great men of the earth. In fact he was not one of the greatest men in the narrower circle of his colleagues in mathematical investigation. But he was a man of fine talent who used the full strength of his powers in his researches; and in his short life (1856–1894) he did excellent work which deserves to be remembered. It is therefore fitting that his articles and memoirs should be brought together in the convenient form of a collected edition. In the two volumes of this work and in the two volumes which record the correspondence between Stieltjes and Hermite we have a complete record* of the scientific activity of Stieltjes and a clear and pleasing insight into his methods of work. The latter is more apparent in the letters, the less formal nature of which gave rise to a more intimate revelation of himself to his friend and later to the world.

His earlier work especially is marked by a careful and deep study of particular questions and the interest which he took in adapting algebraic and analytic formulas to numerical computation. This seems to have been due in considerable measure to the fact that he approached mathematics from the point of view of one engaged in astronomical work and only later gave up what was first conceived to be the object of his scientific life in favor of his studies in mathematics which during the years 1877 to 1883 gained a stronger and stronger grasp upon his thought.

Many of his discoveries were made empirically by incomplete induction from numerous examples developed from the beginning with the delight which he always evinced in numerical computation. In this respect his method has been compared

* See also the section devoted to the (unpublished) "method of Stieltjes" (pp. 357–362, 323) in Poincaré's memoir on the residues of double integrals in *Acta Mathematica*, volume 9, 1887.

with that of Gauss who is known to have discovered experimentally many of his beautiful theorems in the theory of numbers. This method has a peculiar power when it is employed by an intellect of sufficiently keen penetration to divine the general law in the midst of the special properties which belong to particular examples. The procedure of discovery in the case of Stieltjes seems to have been dominated by this empirical method. It seems to be true that nearly all the mathematical truths which he made known were discovered in this way before he was in possession of methods of proof; and that the latter were obtained afterwards by a penetrating analysis of the essential elements of his problem. The truths thus revealed by experiment were subjected to the acid test of logical demonstration and he was mostly free from the enunciation of results for which he had no adequate proof. His letters show the extreme care which he took in the matter of rigorous argumentation. His conversation is said to have shown a wide acquaintance with what may be called interesting mathematical phenomena which his patient calculations had brought to light but which he knew only partially through empirical evidence and not with the clarity and certainty and accuracy of demonstrated truth.

But he seems to have fallen at least once from the high plane of logical precision and accuracy which his published results usually occupied and to have stated a theorem for which he could give no satisfactory proof (though the theorem is probably true). He was engaged (Volume I, page 457) in an investigation of Riemann's celebrated conjecture concerning the distribution of the zeros of the Riemann zeta-function. He transformed the problem so that the truth of the theorem would follow from the convergence of a certain Dirichlet series whose coefficients depended on the function-values of a certain number-theoretic function $g(n)$. This function he examined for a certain property through the ranges of n from 1 to 1200, from 2000 to 2100, and from 6000 to 6100; and, finding the property maintained in these regions for n, he concluded that it was a universal property of $g(n)$. The proof by which he first satisfied himself logically of the existence of the property (see letter 79) he seems never to have published; and the inference to be drawn is that he later found it lacking in some point of accuracy or rigor.

One who reviews the work of Stieltjes at the present time

is spared the necessity of making an analysis of his separate memoirs, for this was done by E. Cosserat soon after the death of Stieltjes in a "Notice" of 62 pages appearing as the opening article of volume 9 (year 1895) of the *Toulouse Annales*. Here we have a résumé of each of the 84 papers now appearing in his Œuvres complètes, with the exception of the last three (which are minor contributions made up by his editors from his unpublished manuscripts). The student of the work of Stieltjes will find these abstracts very valuable for making a rapid survey of the extent of his contributions. [The forty-fifth abstract (and hence the forty-sixth) is perhaps misleading since it quotes the results of Stieltjes without reference to his failure to make good that one of his statements just mentioned in our preceding paragraph.] His great paper (number 80 of the Œuvres) on continued fractions is here discussed only briefly. But one interested in analyzing the contributions made by Stieltjes will certainly wish to examine this paper in full for himself, so that he will not suffer from the absence of a fuller review.

Again, a reviewer at the present time is under no obligation to give a sketch of the life of Stieltjes and relate his scientific investigations to what is thought of usually as the more human aspects of one's career, for this has already been done well by H. Bourget in his "Notice sur Stieltjes" in pages xi to xx of the first volume of the correspondence between Stieltjes and Hermite.

There remains then only the duty to call attention to the features of Stieltjes' work which are of outstanding interest or importance when viewed in the light of present knowledge or which are presumably of particular value to those who now carry the torch of science which he upheld so devotedly with all the strength of his fine talent.

For the purposes of discussion it is convenient to divide the active scientific life of Stieltjes into two parts. The first ends at the time when the influence of the Paris environment and labor and training began to be evident in his work, that is, about 1886 (though he went to Paris in April, 1885). The second period falls in the remaining eight years of his life. The physical bulk of his contributions, exclusive of the 113-page expository paper on the theory of numbers (which falls in the second period), was nearly equal in the two periods; but the two parts are of distinctly unequal merit. Whether

it was the intention of the editors to do so or not I do not know, but they have made the first and second volumes of his Œuvres cover each quite exactly one of these periods in his life, so that the physical division into volumes corresponds closely to a division of his work into distinct parts.

The two parts which may thus be distinguished and separated are yet intimately related and bear throughout the impress of their author's individuality. But the first is given more largely to special problems and shows more clearly the unfolding of the power of Stieltjes' thought, while the latter abounds more in the finished work of his maturer years. Yet in this first volume is to be found the beautiful contribution to the theory of cubic and biquadratic residues, his researches on mechanical quadratures, and his interesting paper on the variation of the density of the earth.

The last three or four years of the first period of his life were particularly full of activity. In this interval he was married, he began to develop what became a life-long intimate friendship with Hermite, and he had the stimulus due to his new abode in Paris. The change and development in the more external affairs of his life were repeated in, or at least had their counterpart in, the remarkable development of his spirit which took place at the same time.

The ingenious conceptions, the generating ideas, the germs of his later activity, multiplied during this interval and in the year or two which followed. To this epoch of his life nearly all his most interesting contributions are to be referred either for their completion or at least for their initial conception. In the few remaining years he developed and extended the ideas and solved some of the problems which had already arisen in his mind.

The centering of the more intensely creative activity of Stieltjes' life so largely in a single short period of it is perhaps instructive. In this period more than in any other new forces moved upon him from without and varied new delights (from new family ties, from the new friendship with Hermite, and from the new opportunity to give all his thought to mathematics) wrought upon his character and life and outlook to make him as it were a new individual. In this period the most of his essentially creative work was done. Is it perhaps true generally that the spirit of man yields its greatest return in new truth discovered when it is subjected to the exhilaration

of the maximum of pleasant change in environment and outlook and subject of interest? This question first arose in my mind and an affirmative answer pressed itself upon my thought when I was once studying the relation of the greater plays of Shakespeare to the character of his life at the time when they were produced; and I have often had occasion to observe a like correspondence in the external life and the more highly creative periods of numerous thinkers in widely separated fields of activity. The work of Stieltjes, while not that of a great master, seems to me nevertheless to be an instructive case in point. If my thesis is well founded it suggests a question of profound value as to how one shall maintain his own thought at the highest possible level of creative exaltation.

It remains to discuss briefly a few of the more important memoirs of the second period in Stieltjes' scientific life. The first of these is his Paris doctor's thesis on certain semi-convergent series. In his work as an astronomer he had often observed the usefulness of certain divergent series (analogous to the celebrated formula of Stirling in the theory of the gamma function) for the purposes of numerical computation and he realized that the theory of these series had never been put on a satisfactory basis. He set himself the task to make a systematic analysis of the matter so that at the end of his study he might be justified in looking upon the series as the asymptotic representation of one or more functions. He studied carefully certain divergent series of special importance and developed their properties so as to obtain from them the most exact information possible as to the numerical values of functions associated with them. He was proceeding largely by his empirical method of gradual approach to the central facts of importance and wide generality; and, if left to himself in this study, he would probably have pushed the investigation to a much wider range. But the genius of Poincaré had turned almost simultaneously and quite independently to this same problem which astronomy had dumbly set before the mathematician for so long a time, and he attacked the problem from a general point of view rather than through special cases; under the fire of his genius the leading secrets were brought to light. Doubtless much remains to be done with the general problem, but the work of Poincaré so overshadowed the simultaneous contribution of Stieltjes that the latter did not pursue the question in further extensive researches; and

his interesting and worthy contribution has largely fallen out of notice.

The most significant among the works of Stieltjes is his celebrated memoir entitled "Recherches sur les fractions continues" (printed on pages 402 to 566 of the second volume of the Œuvres). Here several partial investigations in his earlier work reach their full stature in completed theorems. The germs of some of the most fruitful ideas in the memoir go back as far as the later active years of the first period of his scientific life; and several partial results from his previous scientific activity are woven into it in their proper place and brought to a further stage of completeness than in the earlier papers. In volume 119 of the Paris *Comptes rendus* (at pages 630–632) one will find a report by Poincaré on this memoir. It is described as one of the most remarkable memoirs in analysis "written in the last years," and it is said to place its author in an eminent rank "in the Science of our epoch." In the "Notice" of Bourget we have an interesting, and even a touching, account of the way in which the last discoveries of Stieltjes, which made it possible for him to bring this memoir to completion, so fired the zeal of his spirit for several months as to keep his mind in the freshness of its power even though his last illness was already sapping his physical strength and bringing him to a state of weakness in which he was unable to take up any further labors. He died a few months after the completion of the memoir.

The reputation of Stieltjes can never go higher than the researches recorded in this memoir can take it; and it can never fall lower than the level to which this memoir would bring it; for, though several of his papers contain nothing more than the solutions of problems not inherently difficult, our judgment of the quality of his work will be based primarily on the character of his most worthy effort. In this memoir we find all the qualities of elegance and clarity and marked originality which are characteristic of his better work.

The continued fractions considered by Stieltjes are such that the incomplete quotients are alternately of the forms $a_{2n+1}z$ and a_{2n}, the a_i being real and positive, so that the fraction may be written in the form

$$\frac{1}{a_1 z +} \frac{1}{a_2 +} \frac{1}{a_3 z +} \frac{1}{a_4 +} \cdots.$$

His central result may be stated as follows: If the series Σa_n converges the fraction is oscillatory; the approximating reduced fractions of even order tend to one limit and those of odd order tend to another limit; in each of these two sequences of fractions the numerator and the denominator tend each to an integral function of "genre" zero all of whose roots are real and negative; the limiting form of each of these two sequences from the approximating reduced fractions is a function which is meromorphic throughout the finite plane and is decomposable into a series of simple partial fractions. If the series Σa_n diverges, the continued fraction is convergent and the limit is a function $F(x)$ which is holomorphic everywhere except (possibly) on the negative axis of reals; this function $F(x)$ can be represented by a certain definite integral which in certain cases may be replaced by a series of simple fractions. In connection with the proof of these results Stieltjes derives and utilizes (Chapter V) a remarkable theorem in the general theory of functions, treats (Chapter VI) his celebrated problem of moments and defines (Chapter VI) the now classic Stieltjes integral. An extension of this memoir of Stieltjes has been given by Van Vleck (*Transactions of the American Mathematical Society,* volume 4 (1903), pages 297–332). See also Van Vleck's Boston Colloquium lectures, pages 147–152.

Probably Stieltjes will be longest and most gratefully remembered for his introduction of the integral now called after his name. He employed it in 1894 incidentally to the solution of his continued fractions problem and did not undertake to develop its properties further than was needful for such a use of it. The great importance of the new limiting process was not at once realized and the possibility of its use remained latent for a number of years. But it is more recently coming into its own. Its place is made abundantly clear by Hildebrandt in this BULLETIN, volume 24, 1918, pages 178 ff., in his excellent statement of the reasons why it must be considered a conception of fundamental importance. Perhaps the two strongest are those which arise from the following two results: for every linear functional operation $U(f)$ on continuous functions $f(x)$ there exists a function $u(x)$ of bounded variation such that

$$U(f) = \int_a^b f(x)du(x),$$

the integral being taken in the sense of the definition of Stieltjes; a necessary and sufficient condition that every continuous function on (ab) may be uniformly approximated by linear combinations of a set of functions $[\varphi_1(x), \cdots, \varphi_n(x), \cdots]$ is that the only solution of the equations

$$\int_a^b \varphi_n(x)du(x) = 0, \qquad (n = 1, 2, \cdots)$$

for a $u(x)$ which for every x_0 shall satisfy the condition

$$u(x_0) = \tfrac{1}{2}[u(x_0 + 0) + u(x_0 - 0)]$$

is $u(x)$ = constant.

The way in which Stieltjes came to the introduction of his new limiting process is interesting as illustrating one extreme of the method of discovery. It is as far removed as possible from that method in which one sets out deliberately to extend or generalize conceptions previously found interesting or useful in order to ascertain what further things of value may be seen to grow out from them. Such a method a man of Stieltjes' scientific temperament could never have employed. His mind did not operate in the direction of extending known conceptions by meditating upon them; and apparently he could never have succeeded in working in this way, however well the method may be suited to a certain different and perhaps more robust scientific temperament. On the contrary he proceeded first with particular instances of the problem of a general investigation and solved a number of relatively simple special problems which arose in this way. Being cast down repeatedly by certain difficulties which he could not at first surmount and analyzing the tools which he had employed to ascertain why they did not carry him to the goal, he seems to have come to a realization that the integral which he had been using was not altogether as flexible as the exigencies of his problem demanded. Certain aspects of it he was able to look upon dynamically as a problem in moments. But the moment could not be expressed always in the form of an ordinary integral though it was certainly the limit (in a certain sense) of a sum much like that employed in the Cauchy-Riemann definition of integral. This limit afforded him his new integral. In this way arose one of the highly fruitful conceptions of recent mathematical analysis—one

which will probably play a role of fundamental importance in the further development of certain central branches of mathematics.

R. D. CARMICHAEL.

SHORTER NOTICES.

Opere di Evangelista Torricelli. Edited by GINO LORIA and GIUSSEPPE VASSURA. Faenza, 1919, 2 volumes. Volume I, part 1, xxxviii + 408 pp.; part 2, iv + 482 pp. + plates; Volume II, iv + 322 pp. + plates.

OF those who sat at the feet of Galileo (1564–1642) and from him received instruction and inspiration, two were permitted to enjoy this privilege only in the last weeks of his life. One of these, Viviani (1622–1703), was fifty-eight years his junior and was only twenty years of age when the great teacher passed away. Viviani survived Galileo by sixty-one years, the lives of the two bridging a span of nearly a century and a half. With propriety as well as with pride he could say, in his later life, that he was "postremus Galilei discipulus." In a way, however, Torricelli (1608–1647) could have said the same, for he too was one of the last of those who learned from the great master, although he died so early that he was not, like Viviani, the last disciple to pass away. Viviani signed his famous problem on the hemispherical dome by an anagram of the words "A postremo Galilei Discipulo," while Torricelli was proud to observe that the letters of his own name could be transposed to form the sentence "En virescit Galileus alter."

Of these two great disciples the more brilliant was Torricelli. With a span of life that was less than half as long as that of Viviani, he may be said to have accomplished twice as much, and the results of his labors have been set forth in the volumes under review.

Volume I, consisting of two parts, covers the work of Torricelli in the field of geometry and appears under the editorship of Professor Loria, while Volume II includes his academic lectures, his work in mechanics, and his writings in various minor lines, and is published "per cura" of Professor Vassura.

The introduction to Volume I gives a general survey of the life and works of Torricelli, contains much valuable bibliographical material, and includes considerable interesting information relative to the manuscripts which he left.

The inception of this edition dates from the action of the Congresso Internazionale di Scienze Storiche, held in Rome in 1903. At that time Professor Loria read a paper on "Un' impresa nazionale di universale interesse (pubblicazione delle Opere di Evangelista Torricelli)," with the result that the congress recommended that the Italian government assign to the R. Accademia dei Lincei the task of examining the manuscript works of Torricelli for the purpose of determining what ones should be printed, and also of deciding upon the works already published which should properly find place in a new edition. The recommendation was indorsed by other organizations, and on the occasion of the tercentenary of Torricelli's birth, in 1906, the Consiglio Comunale of Faenza determined to render financial assistance in the publication. The result of this decision was the publication of the present edition, some thirteen years later.

The editors have properly exercised their privilege of rearranging the material so as to present a unified appearance, transferring certain parts of the geometry, which appeared in 1644, to the second volume, and making other changes of a like nature as seemed necessary. This has resulted in a better sequence, and the only criticism that seems proper to make in this respect is that the text does not give, at the beginning of each of the separate divisions, a brief statement of the date and place of the first editions. Such information would be helpful to the bibliophile and historian and would not interfere in any way with the sequence chosen. Information of a somewhat similar nature is given with respect to the material published from hitherto unedited manuscripts.

In his more scientific works Torricelli wrote in Latin, but some of his popular essays and addresses are in Italian. In the present edition no attempt has been made at translation, and the several works appear as in the original. As to sequence, however, many changes have been made. Most of the Opera Geometrica (Florence, 1644) has been included in Volume I, which treats of geometry, but the portion "De motu gravium naturaliter descendentium, et proiectorum libri duo" has been transferred to Volume II, which relates in part

to mechanics. On the other hand, there has been added to the geometry considerable material from the "Discepoli di Galileo" in the manuscripts of the Collezione Galileiana at Florence (volumes 26, 27, 28, 29, 32, 33, and 37). This includes an appendix to lemma XX of the memoir "De dimensione parabolæ," and also a number of other chapters relating to geometry and now for the first time made generally accessible. Among these is Torricelli's Truffle Field ("Campo di tartufi"), with a number of interesting propositions relating chiefly to the circle; his notes "Contro gl'infinito" (mostly in Italian), evidently begun with the idea of placing a series of interesting paradoxes before his students; the essay on the center of gravity of sectors of a circle; the one "De maximis et minimis"; the "Nova per armillas stereometria," containing the particularly interesting chapter "De solidis vasiformis"; the essay "De infinitis spiralibus"; a brief treatment of conic sections; and his essay "De indivisibilibus," a chapter of particular interest to students of the period in which Cavalieri was beginning to pave the way for the calculus which developed towards the end of the seventeenth century. In this connection the reader will also wish to examine the second part of the essay "De centro gravitatis sectoris circuli," where the subject is treated "per geometriam indivisibilium."

The second volume contains the Lezioni Accademiche (in Italian) which were published in Florence in 1715, the essays on mechanics, the heretofore unpublished essay on Prospettiva Pratica (in Italian), and various miscellaneous letters and articles.

The reader who wishes to see Torricelli's classical discussion of the cycloid will find it on page 163 of Volume I, but should also consult the appendix, page 444.

Considering the work as a whole, the general reader will probably be surprised to find that Torricelli, who is generally looked upon as a physicist, wrote chiefly upon geometry. He will find, particularly in the hitherto unpublished chapters, considerable material that can be used with profit in connection with problem courses.

The matter here reprinted from the edition of 1644, judging from several random selections, shows a commendable degree of care on the part of both the editors and the publishers. As would naturally be expected in a work of this size, there are various typographical errors, as in the cases of Huygegs

for Huygens (I, 232), and Withe for White (ibid.). Exception may also be taken to the positive assertion that the birthplace of Thomas White was Hutton, which seems to be only a probability. There are also numerous misprints such as scguento for seguento (I, 294). Matters of this kind, occurring only casually, are too trivial to mention in detail in a review. The most serious defect in the work is the absence of an index, the tables of contents not being sufficiently complete to enable a reader to find easily the particular subject which he wishes to investigate, particularly in connection with the notes.

Aside from the introduction, the work of the editors consists chiefly in the arrangement of the material, with a few important notes such as the one by Professor Loria at the end of the chapter "De tactionibus" (Volume I, page 291). On the whole, the edition is a very satisfactory one, and it is another testimonial to the remarkable scientific and productive powers of Professor Loria.

DAVID EUGENE SMITH.

An Introduction to String Figures. By W. W. ROUSE BALL. Cambridge, W. Heffer and Sons, 1920. 38 pp.

IN the spring of 1920 Mr. Ball gave a lecture at the Royal Institution, London, on simple string figures and their history, and this lecture has now appeared in pamphlet form, designed to set forth, as the title page asserts, "an amusement for everybody." Much of the information given in the essay is already familiar to those who are acquainted (and what student of mathematics is not?) with Mr. Ball's Mathematical Recreations (fifth edition, chapter XVI, page 348), but there is a certain amount of added material in the present publication. On the other hand some of the figures mentioned in the Recreations are not given here. For those who do not have the larger work at hand, this pamphlet will be found of interest.

DAVID EUGENE SMITH.

Solutions of the Examples in a Treatise on Differential Equations. By A. R. FORSYTH. London, Macmillan and Company, 1918. 249 pages.

THIS volume should serve as a time-saver to those who are giving the usual course in differential equations. Since the solution of a differential equation so often depends upon selecting the proper ingenious device, even the experienced

mathematician may find the working out of a particular example to require in some cases a considerable amount of labor. Unless one has already completed a card catalogue or a note book containing his own solutions of such a wide range of examples as is to be found in Forsyth's Treatise, a work like the present will prove of great value.

The examples worked out in the German edition of the Treatise included only those contained in the first and second English editions. As is well known, the later editions contained a great many additional examples. The present volume includes the solution of these. All the examples have been worked out by Professor Forsyth himself, and, with possibly three exceptions, all were found to be solvable in the usual sense of the term.

CHARLES N. MOORE.

Space, Time and Gravitation; an Outline of the General Relativity Theory. By A. S. EDDINGTON. Cambridge, University Press, 1920. vii + 218 pp.

EDDINGTON has a pleasant style even when engaged in technical exposition. His Stellar Movements makes very interesting reading for any mathematician who likes to see what mathematics has recently done toward unraveling the structure of the sidereal universe. This style is a necessity when one tries to write a semi-popular account of Einstein's new theory. On the whole Eddington has succeeded in making the matter clear without appeal to too much mathematics. Of course the person who has absolutely no mathematical outlook beyond the high-school course will have difficulty in appreciating even the Prologue; but let us say a college graduate who has had his calculus, taught in no too formal fashion,—he will find the book possible. The physicist, the not too ignorant philosopher will welcome the chance to study the theory in its elemental simplicity. Einstein's own treatment and that of his followers is about as instructive to the beginner as lectures on generalized (Lagrangian) coordinates would be in collegiate physics as a first treatment of mechanics or Lamé's work as an introduction to the notion of potential.

Eddington begins with a Prologue on What is Geometry written in the form of a Platonic dialogue between an experimental physicist, a pure mathematician, and a relativist.

The three points of view are well done and spiritedly. He should have had a philosopher, but probably he shrank from abandoning his clear and definite style in an attempt to accord or even record such divergent points of view as are taken by Whitehead and Walker as to the philosophic significance of general relativity and whether idealism or realism prevails by virtue of it.*

A discussion of the FitzGerald contraction leads up to relativity (old style, 1905). Then the world of four dimensions (x, y, z, t) is dwelt upon, but no more than is necessary to the uninitiated. The fourth chapter discusses fields of force—the equivalence postulate. Then Kinds of Space serves as the title of a chapter which explains, as well as may be, the meat of non-euclidean geometry, the theory of curvature, the analysis of tensors. A hard chapter to write. (But we wonder whether it is true that Riemann, Christoffel, Ricci and Levi-Civita never dreamt of a physical application of their analysis (page 89). We seem to recall that Ricci, rather early, applied his absolute calculus to problems in elasticity, and Levi-Civita to problems in potential theory. Christoffel is les s known to us, but Riemann we always regarded as physicist quite as much as mathematician, and as a dreamer who dreamt many a relationship between the two.)

It may be well to ponder upon the statement (page 92): "I prefer to think of matter and energy, not as agents causing the degrees of curvature of the world, but as parts of our perceptions of the existence of the curvature." Relative to the older physics this is a turn-about quite Copernican. We have been in the habit of separating our difficulties—space, time, matter, energy, electricity, etc. The new style is to lump them all together and blame it on the way the world (four-dimensional) is made or on the way we make our measurements. It may be the better way. Only the future can tell. Certainly it does not seem the simplification that the Copernican theory represented as against the Ptolemaic. The great advances in science are usually ear-marked by simplicity,—the sort of thing the man in the street can understand if attentive. Einstein is reported to have alleged that not over a dozen in the world could read his book. That is not the way advances in science come,—but this may be

* See "Space, Time and Gravitation," by E. B. Wilson, *The Scientific Monthly*, March, 1920, especially p. 233.

the exception that proves the rule, or Einstein may merely be truthful and we may be fooling ourselves in thinking that the way of scientific advance is natural and easy, once it is pointed out by the pioneer.

Chapter VI is on the New Law of Gravitation and the Old Law. The defects of Newton's law are pointed out and the curvature of the space-time world in the neighborhood of matter is explained in detail with the aid of the differential

$$ds^2 = -dr^2/j - r^2d\theta^2 + jdt^2, \qquad (j = 1 - 2m/r),$$

of the manifold. The diagrams and analogies are suggestive and effective. Next comes a chapter on weighing light and among some excellent statements we find the puzzling one that: Possibly the tails of comets are a witness to the power of the momentum of sunlight which drives outwards the smaller or the more absorptive particles—puzzling until we surmise that probably reflection is not of much importance on account of low albedo. It is in this chapter that an account is given of the results of the eclipse observations at Sobral and Principe. Other tests of the theory form the subject of Chapter VIII,—perihelion of Mercury, minor corrections in the elements of other planetary orbits, shift of the spectral lines, and the philosophical test of the Principle of Equivalence as applied to the clock (page 131) This last argument is particularly interesting if somewhat dangerous, as the author seems to realize.

The chapter on momentum and energy brings in the ordinary relativist law for the change of mass with velocity But the chief interest lies in a quasi-philosophic, speculative attitude wherein gravitation and inertia are identified. If this has appeared in the writings of others, it has at least escaped our attention. There is (page 141) a suggestion that corresponding to any *absolute* property of a volume of a world of four dimensions (the property chosen by Einstein being curvature) there must be four *relative* properties which are conserved (in the Einstein theory, the conservation of energy and of momentum); and that this might be made the starting-point of a general inquiry into the necessary qualities of a permanent perceptional world. The inquiry should be made. Indeed, the statement would seem to place the Einstein theory of gravitation in the class of an astute guess for which such corroboration as the bending of light by definite quantitative

amount and the correct figure for the advance of the perihelion of Mercury could not be expected. It must be possible to modify the curvature by introducing other absolute properties in conjunction with it so as to leave the major fact of (Newtonian) gravitation unaltered but seriously to alter the minor corrections to it. Action, i.e., mass or energy multiplied by time, is given a fundamental position as density multiplied by a four dimensional volume, action as the curvature of the world, and reference is (later) made to the fundamental h of Planck's quantum theory. The curvature of the world in water (density 1 gm./cm.^{3}) is the same as that of space in the form of a sphere of 570,000,000 km. radius or, in time units, of radius about 1/2 hour. A homogeneous sphere of water of this radius would exhaust all space-time. This mass is small compared with many estimates of that of the sidereal universe. Apparently, then, that mass could never get together except in a very much rarer condition than water.

"Towards Infinity" is the title of Chapter X. Here is considered the contrast between translation and rotation. Can an observer accept as a possibility a geocentric system with non-rotating earth? He would have to have something to take care of the centrifugal force $\omega^2 r$. Would a distribution of gravitating masses accomplish this? The author appears to give it up, at least as a problem in physics and to believe there is a real distinction physically if not metaphysically between rotation and translation with respect to absoluteness. (Of course on the Newtonian theory a uniform distribution of negatively gravitating matter about any center will produce a repulsive force proportional to the distance.) A few words are given to acceleration. The curved spaces of de Sitter (spherical) and Einstein (cylindrical) are discussed. The author admits that things are getting speculative and less definite than before and that he is becoming bewildered. Almost all students of generalized relativity must at times share his sensations. One phrase we quote (page 165): The reader will see how our search for an absolute world has been guided by a recognition of the relativity of measurements of physics. This seems sound.

The penultimate chapter on electricity and gravitation gives a sketch of Weyl's theory, a new geometry in which a rod taken around a circuit in space and time changes length, thereby becoming responsible for the sensations of electric

shock. The theory is perplexing but it certainly has some points of philosophic advantage over Einstein's theory, besides leading to the electromagnetic equations. In it *action* is a pure number; there is so and so much *action* in a given region of space-time independent of the coordinates used or the unit of measures. The author thinks that in some way this pure number must be connected with *probability* so that the principle of least action becomes the principle of greatest probability. This is very attractive and, in the present state of our theory, very suggestive. There is a final chapter on the Nature of Things and an Appendix containing mathematical notes.

We agree with Eddington that H. Weyl's Raum, Zeit, Materie (recently appearing in a third edition) is the best treatise on the new relativity; his own is undoubtedly the best general presentation.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
August 15, 1920.

Researches in Physical Optics. Part II: Resonance Radiation and Resonance Spectra. By R. W. WOOD. New York, Columbia University Press, 1919. viii + 184 pp. + X pl.

THERE is no doubt that the theory of spectra of all sorts, including resonance spectra, offers opportunity for mathematical work, both on the side of the dynamics or kinematics of hypothetical atoms or molecules and on the side of empirical nomographic or curve-fitting studies of spectral series. Considerable has been accomplished in both directions but further studies will be necessary before anything approaching satisfaction relative to our knowledge of the intimate parts of optics and of the constitution of matter is reached. Moreover, unless some extraordinary genius like Willard Gibbs appears, to do for this field what he did for physical chemistry, much additional experimental knowledge must be acquired and digested. It is this experimental foundation, with respect to resonance spectra, which Professor Wood is developing and expounding in the researches recently appearing from the Columbia University Press. The book, as written, interests the experimental and descriptive physicist rather than the mathematician, even though he be a mathematical physicist.

E. B. WILSON.

NOTES.

THIS issue of the BULLETIN marks its official transfer to a new Committee of Publication. Since the material for the BULLETIN is prepared far in advance of publication, however, the actual transition will be gradual. Material for the February number, in fact, and some of the material for still later numbers, had been gathered and to some extent arranged by the preceding editors.

The new Committee of Publication, and their associated editors, enter upon their task with mixed feelings and with no better programme than to maintain the high record established in the past. To many it may seem surprising that no acknowledgment of the indebtedness of the Society to those who are now relinquishing control appears in the present number. This defect is explained essentially in the preceding paragraph.

The present editor and his associates bespeak for the BULLETIN under their management the cordial support of contributors and of members of the Society, and forbearance for faults that are inevitable in new and inexperienced direction.

THE seventy-third meeting of the American association for the advancement of science was held at Chicago, December 27 to January 1, under the presidency of Dr. L. O. HOWARD. Professor D. R. CURTISS was vice-president of section A (mathematics). On December 29, Professor O. D. KELLOGG delivered his address as retiring vice-president of the section on "A decade of American mathematics," before a joint session with the Mathematical Association of America and the Chicago section of the American Mathematical Society.

THE concluding (October) number of volume 21 of the *Transactions of the American Mathematical Society* contains the following papers: "Minima of functions of lines," by ELIZABETH LESTOURGEON; "Invariants of infinite groups in the plane," by E. F. SIMONDS; "On triply orthogonal congruences," by J. B. SHAW; "A set of properties characteristic of a class of congruences connected with the theory of functions," by E. J. WILCZYNSKI; "On the equilibrium of a fluid

mass at rest," by J. W. ALEXANDER; "Concerning approachability of simple closed and open curves," by J. R. KLINE.

THE following mathematical papers have appeared in recent volumes of the *Journal of the United States Artillery:* volume 50 (1919): "A method of computing differential corrections for a trajectory," by G. A. BLISS; volume 51: "Numerical integration of differential equations," by F. R. MOULTON; "Equations of differential variations in exterior ballistics," by W. E. MILNE; "The use of adjoint systems in the problem of differential corrections," by G. A. BLISS; "Rotating bands," by OSWALD VEBLEN and P. L. ALGER; "A method of computing differential corrections for a trajectory," by G. A. BLISS; volume 53 (1920): "On weighting factor curves in flat fire," by J. F. RITT; "Mirror and window position finders," by W. C. GRAUSTEIN.

RECENT numbers of the *Proceedings of the National Academy of Sciences* contain: volume 5, number 9 (September, 1919): "Radiation from a moving magneton," by H. BATEMAN; number 12 (December): "Conditions necessary and sufficient for the existence of a Stieltjes integral," by R. D. CARMICHAEL; "Transformations of cyclic systems of circles," by L. P. EISENHART; volume 6, number 1 (January, 1920): "The commutativity of one-parameter transformations in real variables," by A. C. LUNN; number 2 (February): "Groups generated by two operators, s_1, s_2, which satisfy the conditions $s_1^m = s_2^n$, $(s_1s_2)^k = 1$, $s_1s_2 = s_2s_1$," by G. A. MILLER; number 3 (March): "A kinematical interpretation of electromagnetism," by LEIGH PAGE; "Note on geometrical products," by C. L. E. MOORE and H. B. PHILLIPS; number 4 (April): "The starting of a ship," by J. K. WHITTEMORE; "A thermodynamic study of electrolytic solutions," by F. L. HITCHCOCK; "Functionals invariant under one-parameter continuous groups of transformations in the space of continuous functions," by I. A. BARNETT; number 5 (May): "On Kummer's memoir of 1857 concerning Fermat's last theorem," by H. S. VANDIVER; number 6 (June): "On the distortion in conformal mapping when the second coefficient in the mapping function has an assigned value," by T. H. GRONWALL; "On the connection of the specific heats with the equation of state of a gas," by A. G. WEBSTER; "On the conformal mapping of a family of

conics on another," by T. H. GRONWALL; number 7 (July): "On the class number of the field $\Omega(e^{2i\pi/p^n})$ and the second case of Fermat's last theorem," by H. S. VANDIVER; number 9 (September): "On a differential equation occurring in Page's theory of electromagnetism," by H. BATEMAN; "A new proof of a theorem due to Schoenflies," by J. R. KLINE; "On the structure of finite continuous groups with exceptional transformations," by A. C. LUNN.

THE following doctorates in mathematics were conferred by the University of Pairs in 1920: B. GLOBA-MIKHAÏLENKO: "I. Contribution à l'étude des formes d'une masse fluide en équilibre. II. Méthode de Ritz pour l'équilibre, sous l'action d'une charge verticale donnée, d'une lame rectangulaire horizontale encastrée sur tout son contour,"; M. JANET: "I. Sur les systèmes d'équations aux dérivées partielles. II. Sur les congruences de droites."

IN acknowledging the gift to the American Mathematical Society's library of the thirteenth volume of the complete edition of Huygen's works, published by the Dutch Society of Sciences, it was referred to as the final volume of the set (see BULLETIN, volume 27, page 94). This is far from being correct, as several others are expected to appear; in fact, volume 14, published in 1920, has already reached us. This contains his work on the calculus of probabilities, "Van rekeningh in spelen van geluck," 1656–1657, with several appendixes of later date, and various papers on pure mathematics that appeared from 1655 to 1666.

THE Royal society of London has conferred a Royal medal on Professor G. H. HARDY, for his researches in pure mathematics, especially in the analytic theory of numbers.

PROFESSOR A. SOMMERFELD, of the University of Munich, has been awarded the Vahlbruch prize of the Academy of sciences of Göttingen. He has also been elected corresponding member of the Berlin Academy of sciences, and foreign member of the Royal Swedish academy at Upsala.

PROFESSORS A. EINSTEIN and M. PLANCK have been elected corresponding members of the Danish Society of sciences.

PROFESSOR H. WILSKI, of the technical school at Aix, has been elected a member of the Leopoldinisch-Carolinische Akademie at Halle, and Professor E. MÜLLER, of the Vienna technical school, an associate of the Austrian section of that academy.

THE philosophical faculty of the University of Giessen has conferred the honorary degree of doctor of philosophy on Professor C. KOSTKA, of Insterburg.

PROFESSOR E. COHN, formerly of the University of Strassburg, has been appointed honorary professor of theoretical physics at the University of Freiburg.

PROFESSOR C. CRANZ has been appointed professor of theoretical physics at the Charlottenburg technical school.

PROFESSOR E. FISCHER, of the University of Erlangen, has been appointed professor of mathematics at the University of Cologne.

AT the Dresden technical school, Professor G. KOWALEWSKI, of the German University at Prague, has been appointed professor of mathematics, Dr. M. LAGALLY, of Munich, professor of applied mathematics, and Dr. L. FÖPPL, of the University of Würzburg, professor of the theory of stability, hydrodynamics, and aerodynamics.

DR. R. GRAMMEL, of the University of Halle, has been appointed to a professorship at the Stuttgart technical school.

DR. W. LIETZMANN has been appointed professor of the teaching of the exact sciences at the University of Göttingen.

PROFESSOR H. LIEBMANN has been appointed professor of mathematics at the University of Heidelberg.

ASSOCIATE professor H. ROTHE, of the Vienna technical school, has been promoted to a full professorship of mathematics.

PROFESSOR E. STEINITZ, of the Breslau technical school, has been appointed professor of mathematics at the University of Kiel.

PROFESSOR R. WEITZENBÖCK, of the German technical school at Prague, has been appointed professor of mathematics at the technical school at Graz.

PROFESSOR A. FÖPPL, of the Munich technical school, has retired from active teaching.

THE following persons have been admitted as privat-docents: Dr. J. NIELSEN, for pure and applied mathematics, at the University of Hamburg; Dr. O. v. GRUBER, for applied mathematics, at the Munich technical school; Dr. G. WIARDA, for mathematics, at the University of Marburg.

PROFESSOR J. HADAMARD has been appointed professor of mathematical analysis at the Ecole centrale des arts et manufactures at Paris.

M. CERF has been appointed associate professor of mathematics at the University of Dijon.

DR. C. G. KNOTT has been appointed to the newly instituted office of reader in applied mathematics at the University of Edinburgh.

MR. F. J. HARLOW, head of the department of mathematics and physics at the Sir John Cass Technical Institute, Aldgate, has been appointed principal of the Municipal Technical College, Blackburn.

MISS NORA I. CALDERWOOD and MR. T. A. LUMSDEN have been appointed assistant lecturers in mathematics at the University of Birmingham.

AT the University of Nebraska, Dr. T. A. PIERCE has been promoted to an assistant professorship of mathematics, Mr. W. M. BOND, Mr. O. C. COLLINS, of Oxford University, and Mr. C. R. SHERER have been appointed instructors, and Miss C. RUMMONS assistant instructor.

IN the department of mathematics at the University of Illinois, associate professor R. D. CARMICHAEL has been promoted to a full professorship; Dr. C. F. GREEN, Dr. L. L. STEIMLEY, and Dr. B. MARGARET TURNER have been appointed instructors; Professor E. R. SMITH, on leave of absence from Pennsylvania State College, has been appointed associate.

DR. LUDWIK SILBERSTEIN, author of works on mathematical physics, and formerly lecturer at the University of Rome, has accepted a research position with the Eastman Kodak Company at Rochester.

PROFESSOR HARRIET GLAZIER, on leave of absence from the Western College for Women, has been appointed professor of mathematics at the Southern Branch of the University of California for the current academic year, during the absence of Professor MYRTIE COLLIER.

AT the University of Pennsylvania, assistant professor F. H. SAFFORD has been promoted to a full professorship of mathematics, and Mr. H. M. GEHMAN and Mr. R. W. HARTLEY have been appointed instructors. Dr. R. A. ARMS has resigned, to accept a professorship at Pennsylvania College, Gettybsurg.

IN the department of mathematics at the United States Naval Academy, assistant professors J. A. BULLARD, A. DILLINGHAM, and J. N. GALLOWAY have been promoted to associate professorships, and Mr. M. A. EASON, Mr. H. H. GAVER, Mr. H. M. ROBERT, Jr., and Dr. W. F. SHENTON to assistant professorships; Mr. E. A. BAILEY, Mr. A. J. BARRETT, Mr. E. R. C. MILES, and Mr. A. A. ROBINSON have been appointed instructors.

MR. OSCAR SCHMIEDEL has been appointed professor of mathematics at Nebraska Wesleyan University.

DR. O. J. RAMLER has been promoted to an associate professorship of mathematics at the Catholic University of America.

PROFESSOR H. S. MYERS, of Huron College, has been appointed professor of mathematics at Southwestern College, Winfield, Kansas.

PROFESSOR R. A. WELLS, of Park College, has been appointed associate professor of mathematics at the Michigan State Normal College.

MR. CORNELIUS GOUWENS, of the University of Kansas, has been appointed assistant professor of mathematics at Iowa State College.

PROFESSOR C. E. HORNE, of the University of Porto Rico, has been appointed dean of the college of agriculture and mechanic arts at the university of Mayagüez, P. R.

ASSISTANT professor F. C. KENT has been promoted to an associate professorship of mathematics at the Oregon Agricultural College.

MR. H. L. OLSON, of the University of Wisconsin, and Mr. H. A. SIMMONS have been appointed instructors in mathematics at the University of Michigan.

AT the University of Alberta, Mr. T. H. MILNE, of the University of Toronto, has been appointed lecturer, and Mrs. E. T. MITCHELL instructor in mathematics; Mr. GEORGE ROBINSON has resigned, to become assistant to Professor E. T. WHITTAKER in his mathematical laboratory at the University of Edinburgh.

MISS ANNA M. MULLIKIN and Miss HELMA HOLMES have been appointed instructors in mathematics at the University of Texas.

MR. C. E. HARRINGTON has been appointed instructor in mathematics at the University of Buffalo.

MR. L. M. GRAVES has been appointed instructor in mathematics at Washington University.

DR. A. R. WILLIAMS and Dr. C. D. SHANE have been appointed instructors in mathematics at the University of California.

PROFESSOR M. KRAUSE, of the Dresden technical school, died March 2, 1920, at the age of sixty-eight.

PROFESSOR L. POCHHAMMER, of the University of Kiel, died March 24, 1920, at the age of seventy-eight.

THE death is announced of the distinguished geometer Professor K. F. W. ROHN, of the University of Leipzig, on August 4, 1920, at the age of sixty-five.

THE death is announced of Professor E. SELLING, formerly of the University of Würzburg.

ASSISTANT professor T. R. DAVIES, of McGill University, died August 12, 1920, at the age of fifty-six.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALLY (E.). Géométrie synthétique des unicursales de troisième classe et de quatrième ordre. Paris, Gauthier-Villars, 1920. 6 + 100 pp.

BEUTEL (E.). Die Quadratur des Kreises. 2te Auflage. (Mathematische-physikalische Bibliothek, Nr. 12.) Leipzig, Teubner, 1920. 57 pp.

CAJORI (F.). A history of the conceptions of limits and fluxions in Great Britain from Newton to Woodhouse. Chicago and London, Open Court, 1919. 300 pp. $2.00

CORPUT (J. G. VAN DER). Over roosterpunten in het platte vlak. (De beteekenis van de methoden van Voronoï en Pfeiffer.) (Akademisch Proefschrift.) Groningen, Noordhoff, 1919.

DICKSON (L. E.). History of the theory of numbers. Volume 2: Diophantine analysis. Washington, Carnegie Institution, 1920. 16 + 803 pp.

FEHR (H.). See LEVEUGLE (R.).

FUBINI (G.). Lezioni di analisi matematica. 3a edizione interamente rifusa. Torino, Società tipografico editrice nazionale, 1919. 8vo 8 + 512 pp.

GALE (A. S.) and WATKEYS (C. W.). Elementary functions and applications. New York, Holt, 1920. 20 + 436 pp. $2.60

LEBON (E.). Table de caractéristiques de base 30,030 donnant, en un seul coup d'œil, les facteurs premiers des nombres premiers avec 30,030 et inférieurs à 901,800,900. Tome I, 1er fascicule. Paris, Gauthier-Villars, 1920. 4to. 56 pp.

LEVEUGLE (R.). Précis de calcul géométrique. Avec une préface de H. Fehr. Paris, Gauthier-Villars, 1920. 8vo. 56 + 400 pp.

MACMAHON (P. A.). An introduction to combinatory analysis. Cambridge, University Press, 1920. 8vo. 6 + 71 pp. 7s. 6d.

MANSION (P.). Derniers mélanges mathématiques. Paris, Gauthier-Villars, 1920. 8vo. 4 + 188 pp. Fr. 10.00

ROUGIER (L.). Le philosophie géométrique d'Henri Poincaré. Paris, Alcan, 1920. 8vo. 208 pp. Fr. 9.00

VOOREN (W. L. VAN DER). Grenswaarden, eene inleiding tot de differentiaal- en integraalrekening. Groningen, Noordhoff, 1919. 8vo. 89 pp.

WATKEYS (C. W.). See GALE (A. S.).

WHITEHEAD (A. N.). The concept of nature. Tarner lectures delivered in Trinity College November 1919. Cambridge, University Press, 1920. 8vo. 10 + 202 pp.

II. ELEMENTARY MATHEMATICS.

BOREL (E.). Die Elemente der Mathematik. Vom Verfasser genehmigte deutsche Ausgabe besorgt von P. Stäckel. 2ter Band: Geometrie. 2te Auflage. Leipzig, Teubner, 1920. 16 + 380 pp.

CRANTZ (P.). Sphärische Trigonometrie zum Selbstunterricht. (Aus Natur und Geisteswelt, Nr. 605.) Leipzig, Teubner, 1920. 98 pp.

GRAY (J. C.). Number by development. Volume 3: Grammar grades. Philadelphia, Lippincott, 1919. 20 + 514 pp.

GUGLE (M.). Modern junior mathematics. Book 3. New York, Gregg, 1920. 13 + 246 pp. $1.00

HEDRICK (E. R.). Logarithmic and trigonometric tables. Revised edition. New York, Macmillan, 1920. 22 + 143 pp.

HJELMSLEV (J.). Elementær Geometri. 2den Bog. København, Gjellerup, 1919. 8vo. 111 pp.

KÖHLER (A.). Methodischer Führer und Ratgeber für den mathematischen Unterricht. Teil II: Planimetrie, Trigonometrie und Stereometrie. Stettin, Fischer und Schmidt, 1919. 395 pp.

NEUFELD (J. L.). Elementary algebra with a table of logarithms. Philadelphia, Blakiston, 1920. 12 + 383 pp.

STÄCKEL (P.). See BOREL (E.).

III. APPLIED MATHEMATICS.

ANDRIEU (—.). Les révélations du dessin et de la photographie à la guerre. Principes de métrographie. Paris, Gauthier-Villars, 1920. 8vo. 4 + 138 pp. Fr. 16.00

BAYLE (G.). Statique graphique. 2e édition. Paris, Librairie de l'Enseignement technique, 1920. 8vo. 164 pp. Fr. 15.00

BLOCH (W.). Einführung in die Relativitätstheorie. 2te, verbesserte Auflage. (Aus Natur und Geisteswelt, Nr. 618.) Leipzig, Teubner, 1920. 106 pp.

BUCHANAN (D.). See MOULTON (F. R.).

BUCK (T.). See MOULTON (F. R.).

CARL (A.). See KRAUSE (M.).

CHWOLSON (O. D.). Lehrbuch der Physik. 2ter Band, 1te Abteilung: Die Lehre vom Schall. Herausgegeben von G. Schmidt. 2te Auflage. Braunschweig, Vieweg, 1919.

DUBOSQUE (J.). Etudes théoriques et pratiques sur les murs de soutènement et les ponts et viaducs en maçonnerie. 6e édition. Paris et Liège, Béranger, 1920. 8vo. 380 pp. Fr. 48.00

EGERER (H.). Lehrbuch der technischen Mechanik in vorwiegend graphischer Behandlung. 1ter Band: Graphische Statik starrer Körper. Berlin, Springer, 1919. M. 20.00

Föppl (A.) und Föppl (L.). Drang und Zwang. Eine höhere Festigkeitslehre für Ingenieure. 1ter Band. München, Oldenburg, 1920.

Föppl (L.). See Föppl (A.).

Freundlich (E.). Die Grundlagen der Einsteinschen Gravitationstheorie. 3te Auflage. Berlin, Springer, 1920. 96 pp. Geh. M. 6.80

Gerday (C.). See Moulan (P.).

Griffin (F. L.). See Moulton (F. R.).

Hahn (K.). Grundriss der Physik. Leipzig, Teubner, 1920. 8vo. 330 pp.

Huyghens (C.). Traité de la lumière. (Les Maîtres de la Pensée scientifique.) Paris, Gauthier-Villars, 1920. 18mo. 10 + 155 pp. Fr. 3.60

Krause (M.). Analysis der ebenen Bewegung. Unter Mitwirkung von A. Carl. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1920.

Loewy (A.). Mathematik des Geld- und Zahlungsverkehrs. Leipzig, Teubner, 1920. 8 + 273 pp.

Longley (W. R.). See Moulton (F. R.).

MacMillan (W. D.). See Moulton (F. R.).

Marcelin (A.). Propagation et réception des sons dans l'eau. (Thèse, Paris.) Paris, Imprimerie Lahure, 1920. 8vo. 92 pp.

Moulan (P.). Cours de mécanique élémentaire à l'usage des écoles industrielles. 4e édition, revue et augmentée par C. Gerday. Paris et Liège, Béranger, 1920. 8vo. 1291 pp. Fr. 40.00

Moulton (F. R.). Periodic orbits. By F. R. Moulton, in collaboration with D. Buchanan, T. Buck, F. L. Griffin, W. R. Longley, and W. D. MacMillan. (Carnegie Institution of Washington, Publication No. 161.) Washington, Carnegie Institution, 1920. 4to. 16 + 524 pp.

Pigeaud (G.). Résistance des matériaux et élasticité. Paris, Gauthier-Villars, 1920. 8vo. 16 + 772 pp. Fr. 64.00

Rose (W. N.). Mathematics for engineers. Part 2. London, Chapman and Hall, 1920. 8vo. 14 + 419 pp. 13s. 6d.

Schmidt (G.). See Chwolson (O.).

Adolfo Stahl lectures in astronomy. Delivered at San Francisco by members of the staffs of the Lick Observatory, the Mount Wilson Solar Observatory, and the Students Observatory at Berkeley. Published by the Astronomical Society of the Pacific. Stanford University Press, 1919. $2.75

Tresling (J.). Deformaties en trillingen in het vaste lichaam bij afwijkingen van de wet van Hooke, ook in verband met de toestandsvergelijking. Leyden, E. Ydo, 1919. 8vo. 5 + 83 pp.

Vries (H. de). Leerboek der beschrijvnde meetkunde. Eerste deel. Tweede druk. Delft, Technische Boekhandel en Drukkerij, J. Waltman, Jr., 1919.

THE THIRTEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

The thirteenth regular meeting of the Southwestern Section of the American Mathematical Society was held at the University of Nebraska at Lincoln, on Saturday, November 27, 1920. About twenty persons attended the meeting, including the following members of the Society:

Professor C. H. Ashton, Professor W. C. Brenke, Professor A. L. Candy, Professor C. A. Epperson, Professor M. G. Gaba, Professor E. R. Hedrick, Professor Louis Ingold, Professor T. A. Pierce, Professor H. L. Rietz, Professor W. H. Roever, Professor Oscar Schmiedel, and Professor E. B. Stouffer.

On Friday evening, before the meeting, a smoker was given for the visiting members and their friends, in the University Temple. At this gathering, Professor Hedrick spoke informally concerning the reports of the National Committee on Mathematical Requirements. An interesting discussion followed this informal address. A feature much appreciated by those present was the exhibit of rare mathematical manuscripts and textbooks, by Professor T. J. Fitzpatrick.

At the business session it was decided to hold the next meeting of the Section at the University of Missouri at Columbia. The following committee was appointed:
Professor E. R. Hedrick (Chairman), Professor W. C. Brenke, and Professor E. B. Stouffer (Secretary).

The following papers were presented:

(1) Professor H. J. Ettlinger: "Existence and oscillation theorems for a system of n differential equations of the second order."

(2) Professor Otto Dunkel: "The curve which with its caustic encloses the minimum area."

(3) Professors E. R. Hedrick, Louis Ingold, and W. D. A. Westfall: "Classification of infinities of transformations."

(4) Professor W. C. Brenke: "On the convergence of certain types of infinite determinants."

(5) Professor Louis Ingold: "Rotation formulas and invariants."

(6) Professor T. A. Pierce: "Note on Bernoulli's numbers."

(7) Professor Oscar Schmiedel: "Summation of certain infinite series in finite form. Preliminary report."

(8) Professor M. G. GABA: "A set of postulates for line geometry in terms of line and transformation."

(9) Professor E. B. STOUFFER: "Calculation of the invariants and covariants for ruled surfaces."

(10) Professor E. B. STOUFFER: "Semi-covariants of a general system of linear homogeneous differential equations."

(11) Professor H. L. RIETZ: "Frequency distributions that result from appyling certain transformations to normally distributed variates."

(12) Professor S. LEFSCHETZ: "On certain invariant numbers of algebraic varieties, with application to abelian varieties."

In the absence of the authors, the papers of Professor Ettlinger, Professor Dunkel, and Professor Lefschetz were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Ettlinger considers the system

$$\frac{d}{dx_i}\left[K_i(x_i, \lambda_1, \lambda_2, \cdots, \lambda_n)\frac{du}{dx_i}\right] - G_i(x_i, \lambda_1, \lambda_2, \cdots, \lambda_n)u = 0$$

$$A_{ij}u_i(a_i) - B_{ij}K_i(a_i, \lambda_1, \lambda_2, \cdots, \lambda_n)\frac{du_i(a_i)}{dx_i}$$
$$\equiv C_{ij}u_i(b_i) - D_{ij}K_i(b_i, \lambda_1, \lambda_2, \cdots, \lambda_n)\frac{du_i(b_i)}{dx_i},$$

where A_{ij}, B_{ij}, C_{ij}, D_{ij} are functions of $\lambda_1, \lambda_2, \cdots, \lambda_n$, and where $i = 1, 2, \cdots n$ and $j = 1, 2$. Under suitable restrictions, the existence of characteristic numbers for the system is proved, and an oscillation theorem established.

2. Necessary conditions for the minimum area between a curve and its caustic were given by Professor P. R. Rider in a paper read at the April meeting of the Chicago Section of the Society. In this paper Professor Dunkel derives conditions for the minimum area which are necessary and sufficient. The properties of the minimizing curve and its caustic are studied, and it is shown that in the general case the first has a single cusp while the second has two, and that the three cusps have parallel tangents. Several cases of end conditions are examined.

3. In this paper Professors Hedrick, Westfall and Ingold continue the investigation of transformations of which an earlier portion was given in a paper at the summer meeting of the Society in Chicago. A classification of the infinities of a transformation is made, and their principal characteristics are discussed, in a manner analogous to the corresponding matter in the theory of functions of a complex variable.

4. It can be shown easily that the infinite determinant $D(a)$ converges to the value zero if the absolute values of its elements a_{ik} form a convergent series. Professor Brenke shows that this remains true if the elements have the form $\lambda + a_{ik}$, where λ is a constant and $\Sigma|a_{ik}|$ converges; also that the infinite secular determinant, whose principal diagonal elements are $x + a_{ii}$, converges for $|x| \leqq 1$, the value being zero for $|x| < 1$.

5. In this paper Professor Ingold extends the application of the formulas of a former paper on "Rotations in a function space" and shows that certain of these formulas give rise to important invariants. The case of the rotations of a system of two vectors is treated in considerable detail.

6. In his note Professor Pierce sets up an explicit formula for the nth Bernoulli number and uses this formula for determining certain factors of the denominators of Bernoulli's numbers.

7. In this paper Professor Schmiedel deals with series principally of the type $1 + 2^a x + 3^a x^2 + \cdots + (m + 1)^a x^m$, and seeks to find expressions for the sum valid for all integral values of a and m. Two expressions for each sum are found according as m is positive or negative, when one or the other of the series is convergent. The special case $x = 1$ is not excepted.

Consideration of the sum for a negative being withheld, the paper is given as a preliminary report.

8. Professor Gaba takes for his basis an undefined element called line and an undefined operation on lines called transformation. In terms of line and transformation point and plane are defined. His six postulates, which are theorems of

line geometry, are sufficient to prove as theorems the postulates of Hedrick and Ingold or of Veblen and Young. Independence examples for the postulates are given.

9. In Wilczynski's Projective Differential Geometry of Curves and Ruled Surfaces, calculation is made of a system of invariants and covariants associated with non-developable ruled surfaces. In the present paper, Professor Stouffer obtains the same results by methods involving much less labor. The general system of two linear homogeneous differential equations of the second order is first reduced to a canonical form, the invariants and covariants of this simplified form are then calculated, and the results thus obtained are finally expressed in terms of the coefficients and variables of the original system of differential equations.

10. In a paper read before the Society in April, 1920 (see abstract in this BULLETIN for June, 1920) Professor Stouffer calculated a complete system of seminvariants of a general system of linear homogeneous differential equations of the mth order in n variables. Those results are now extended by the calculation of a complete system of semi-covariants of the same general system of differential equations.

11. In this paper, Professor Rietz gives the results of an investigation into the nature of the frequency distributions that result from applying certain transformations to each variate of a normal distribution. The transformations used are suggested by experience with various systems of variates, some of which are measurements of distances, others of surfaces, and others of volumes. In particular, if the transformation replaces each variate x by a variate $z = kx^n$, where k and n are constants, the frequency function that gives the distribution of z's is found, and some of its interesting properties are established for special values of n.

12. Professor Lefschetz's paper will appear in an early number of the *Transactions*.

LOUIS INGOLD,
Acting Secretary of the Section.

A REMARK ON SKEW PARABOLAS.

BY PROFESSOR GINO LORIA.

(Extract from a letter to Professor D. E. Smith.)

Permettez-moi une remarque relative à deux articles publiés dans le BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY.

Dans le cahier d'Octobre 1918 Mary F. Curtis a établi le remarquable résultat, que toute parabole gauche rectifiable est une hélice; elle s'est servi des formules suivantes pour representer une des courbes dont il s'agit:

$$x = at^3, \qquad y = bt^2, \qquad z = ct; \tag{1}$$

en conséquence l'axe Oz est le tangente à la courbe à l'origine des coordonnées, tandis que le plan $x = 0$ est le plan osculateur; l'axe Oy est donc la normale principale de la courbe et Ox la binormale. Les formules (1) s'appliquent donc à toute parabole gauche, les axes coordonnées étant orthogonaux.

Or tout celà est échappé à M. Hayashi, qui, un an après (BULLETIN, Octobre 1919) a crû nécessaire de traiter la même question de nouveau à l'aide d'une représentation plus compliquée de le (1), ce qui l'a entrainé à des calculs plus longs de ce qu'il est nécessaire. Cette remarque me parait utile, car les calculs du mathématicien japonnais pourraient faire croire que les formules (1), *les axes étant rectangulaires*, sont applicables *seulement à une classe* de paraboles gauches, tandis qu'on a droit de les *appliquer à toutes*.

Je vous authorise, cher Monsieur, de faire de tout celà, l'usage qui vous semble bon.

GENOA, ITALY,
November 13, 1920.

THE PSEUDO-DERIVATIVE OF A SUMMABLE FUNCTION.

BY PROFESSOR WILLIAM L. HART.

(Read before the American Mathematical Society September 8, 1920.)

Introduction.—In the present paper we shall consider two different types of derivative functions, to be termed *pseudo-derivatives.* These derivatives will be defined for real-valued functions satisfying suitable conditions of summability. The definitions will state certain functional relations which the pseudo-derivatives must satisfy. These relations involve the well-known notion of convergence in the mean, explained in § 1, below. The pseudo-derivative of the first type is considered in § 2, and that of the second type in § 3. The present paper is concerned merely with the definition of these derivatives and with a discussion of a few of their properties. Applications of such derivatives will be considered in a later paper.

All integrals in this paper are taken in the Lebesgue sense. A function f will be termed *integrable* if both f and f^2 are summable in the Lebesgue sense. All functions and variables will be supposed real-valued. The phrase *almost everywhere* will mean *with the exception of at most a set of points of measure zero.* In the discussion below two functions $F_1(x)$ and $F_2(x)$, defined on a set E, will be called the same if they are equal almost everywhere on E. In all equations below, as for example $F_1(x) = F_2(x)$, it will be understood that the equality may cease to hold on some sub-set of E of measure zero.

1. *Convergence in the Mean.*—We shall have occasion to use a slight extension of the notion of convergence in the mean as it has been defined in the case of sequences of functions.* Let $q(s, h)$ be defined for $|h| \leqq h_0$, $h \neq 0$, and for s on a measurable set E, where s may be considered as a single or as an m-partite variable $(s_1, s_2, \cdots, s_m)$. For every h, suppose that $q(s, h)$ is integrable on E.

DEFINITION 1. *The function $q(s, h)$ converges in the mean*

* Cf. Plancherel, *Rendiconti del Circolo Matematico di Palermo*, vol. 30 (1910), p. 292.

to an integrable function $g(s)$ *as* h *approaches zero in case*

$$\lim_{h=0} \int_E [q(s, h) - g(s)]^2 ds = 0. \tag{1}$$

We shall say that $g(s)$ is the *limit in the mean* and shall write, for abbreviation,

$$\underset{h=0}{\text{l.m.}}\ q(s, h) = g(s). \tag{2}$$

This type of convergence has properties similar to those* possessed by convergence in the mean as defined for sequences. Certain properties of $g(s)$ are listed below; the proofs of the statements will not be given because of their obvious relationship to proofs in the case of sequences of functions.

(*a*) A necessary and sufficient condition that (1) should hold for some function $g(s)$ is that

$$\lim_{h_1, h_2=0} \int_E [q(s, h_1) - q(s, h_2)]^2 ds = 0.$$

(*b*) If $g(s) = g'(s)$ and $g(s) = g''(s)$ both satisfy (2), then $g'(s) = g''(s)$.

(*c*) At least one sequence of values $(h_1, h_2, \cdots, h_n, \cdots)$, with $\lim_{n=\infty} h_n = 0$, can be selected so that $\lim_{n=\infty} q(s, h_n) = g(s)$, almost uniformly† for s on E.

(*d*) The function $g(s)$ satisfies the equation

$$\lim_{h=0} \int_E q^2(s, h) ds = \int_E g^2(s) ds. \tag{3}$$

(*e*) Every sequence $(h_1, h_2, \cdots)$ with $\lim_{n=\infty} h_n = 0$, corresponding to which the sequence $[q(s, h_n);\ n = 1, 2, \cdots]$ converges almost everywhere on E, has the property that

$$\lim_{n=\infty} q(s, h_n) = g(s),$$

almost everywhere on E.

2. *The Pseudo-derivative of the First Kind.*—Let $f(x)$ be defined and integrable for values of x on some measurable set E and let $q(x, h) = [f(x + h) - f(x)]/h$. For a given value of h, if $(x + h)$ is not in E, define $f(x + h) = f(x)$. If, for a given h and x, $f(x + h)$ and $f(x)$ are infinities of the same sign, let

* Cf. Plancherel, loc. cit., p. 294.

† Cf. Plancherel, loc. cit., p. 292.

$q(x, h)$ be given any arbitrary value, say zero. Then, for every $h \neq 0$, $q(x, h)$ is integrable on E.

DEFINITION 2. *The pseudo-derivative of* $f(x)$ *is* $\text{l.m.}_{h=0}\, q(x, h)$, *provided this limit exists.* It will be denoted by the symbol $f_x(x)$.

As a consequence of Definition 1, $f_x(x)$ is unique if it exists, and satisfies the functional equation

$$\lim_{h=0} \int_E [q(x, h) - f_x(x)]^2 dx = 0. \tag{4}$$

Moreover, at least one sequence $(h_1, h_2, \cdots)$ can be selected as in (c), § 1, such that

$$\lim_{n=\infty} q(x, h_n) = f_x(x), \tag{5}$$

almost uniformly for x on E. Suppose, for example, that E is the set of all irrational numbers on the interval (0, 1) and let $f(x) = x$. Then it is easy to show that $f_x(x) = 1$.

In analogy with the procedure followed in dealing with the classical derivative, let us denote the pseudo-derivative of order n of $f(x)$ by the symbol $f_{x^n}(x)$, and let us define it to be the pseudo-derivative of $f_{x^{n-1}}(x)$, for $n = 2, 3, \cdots$. In the case of a function $f(x; y_1, \cdots, y_k)$, let us define $f_x(x; b_1, \cdots, b_k)$, the partial pseudo-derivative with respect to x at the point $(y_1 = b_1, \cdots, y_k = b_k)$, to be the pseudo-derivative in the sense of Definition 2 of the function $f(x; b_1, \cdots, b_k)$.

In the two theorems which follow, conditions are given under which the pseudo-derivative is equal to the classical derivative.

THEOREM I. *Suppose that the pseudo-derivative* $f_x(x)$ *exists and that* $f(x)$ *also possesses a derivative* $df(x)/dx$ *almost everywhere on* E. *Then* $df(x)/dx = f_x(x)$ *almost everywhere on* E.

Since $df(x)/dx$ exists, we have the equation

$$\lim_{h=0} q(x, h) = \frac{df(x)}{dx}, \tag{6}$$

almost everywhere on E. Since (5) and (6) both hold almost everywhere on E, it follows that the theorem is true.

THEOREM II. *Let the set* E *be the closed interval* (a, b) *and suppose that a constant* $K > 0$ *exists such that, for all values of* $h \neq 0$ *and for all points* x *on* (a, b), $|q(x, h)| \leqq K$. *Then,*

if the classical derivative exists at each point of (a, b), *the pseudo-derivative* $f_x(x)$ *exists and* $f_x(x) = df(x)/dx$.

For values $x > b$ and $x < a$, define $f(x)$ by the equations

$$f(x) = f(a) + (x - a)\frac{df(a)}{dx} \quad (x < a),$$

$$f(x) = f(b) + (x - b)\frac{df(b)}{dx} \quad (x > b).$$

Since $f(x)$ is measurable, it is seen that $df(x)/dx$ is also measurable. Moreover, as a consequence of a well-known property* of Lebesgue integrals it is allowable to invert $\lim_{h=0}$ and $\int_{(ab)}$ on the left side of the following equation and to obtain

$$\lim_{h=0}\int_{(ab)}\left[q(x, h) - \frac{df(x)}{dx}\right]^2 dx = 0.$$

Hence, by definition, $f_x(x) = df(x)/dx$.

It is easily verified that Theorem II remains true if, in place of the assumption of the theorem in regard to $df(x)/dx$, we assume that it exists everywhere on (a, b) except at points x on a set H of measure zero.

3. *The Pseudo-derivative of the Second Kind.*—Let us consider a function $f(x, s)$ defined for values of the variable x on a closed interval (a, b) and for values of the m-partite variable $s = (s_1, \cdots, s_m)$ on a measurable set E. In all the discussion below suppose that, for every x on (a, b), $f(x, s)$ is integrable on E and let

$$(7) \qquad q(x, h, s) = \frac{f(x + h, s) - f(x, s)}{h}.$$

For a given value of $h \neq 0$, if the point $(x + h)$ is not on (a, b), define $f(x + h, s) = f(x, s)$. For points s at which $f(x + h, s)$ and $f(x, s)$ are infinities of the same sign, let us give $q(x, h, s)$ any arbitrary value, say zero. Then, if $x = x_0$ is on (a, b), it follows that the function $q(x_0, h, s)$, for every $h \neq 0$, is integrable on E.

DEFINITION 3. *The pseudo-derivative of the second kind of* $f(x, s)$ *with respect to* x *at the point* $x = x_0$, *is* $\text{l.m.}_{h=0}q(x_0, h, s)$, *provided this limit exists.* It will be denoted by the symbol $f_x'(x, s)$.

* Cf. de la Vallée Poussin, *Intégrales de Lebesgue*, p. 44.

For brevity, the designation "second kind" will be omitted in the future in this section of the paper in speaking of the pseudo-derivatives $f_x'(x, s)$. As a consequence of Definition 1, if the function $f_x'(x_0, s)$ exists, it is unique and satisfies the functional equation

$$(8) \qquad \lim_{h=0} \int_E [q(x_0, h, s) - f_x'(x_0, s)]^2 ds = 0.$$

If $f(x, s)$ is constant with respect to s it is seen that $f_x'(x, s)$ reduces to the classical derivative of a function of the single variable x. Moreover, from (c), § 1, it follows that at least one sequence $(h_n;\ n = 1, 2, \cdots)$ can be selected with $\lim_{n=\infty} h_n = 0$, such that

$$(9) \qquad \lim_{n=\infty} q(x_0, h_n, s) = f_x'(x_0, s),$$

almost uniformly for s on E.

In the theorems that follow some useful properties of the pseudo-derivative $f_x'(x, s)$ are considered.

THEOREM III. *Suppose that at $x = x_0$ the pseudo-derivative $f_x'(x_0, s)$ exists, and that at $x = x_0$ the partial derivative (in the classical sense) $\partial f(x_0, s)/\partial x$ exists, finite or infinite, almost everywhere on E. Then it follows that $\partial f(x_0, s)/\partial x = f_x'(x_0, s)$.*

By hypothesis we have, almost everywhere on E,

$$(10) \qquad \lim_{h=0} q(x_0, h, s) = \frac{\partial f(x_0, s)}{\partial x}.$$

It is easily verified that the theorem is an immediate consequence of equations (9) and (10).

THEOREM IV. *Let the function $p(s)$ be bounded and measurable on E, and let $F(x, s) = f(x, s)p(s)$. Then, if $f_x'(x_0, s)$ exists, it follows that there exists, likewise, $F_x'(x_0, s) = f_x'(x_0, s)p(s)$.*

The existence of $F_x'(x_0, s)$ follows from the fact that

$$\lim_{h=0} \int_E [q(x_0, h, s)p(s) - f_x'(x_0, s)p(s)]^2 ds = 0,$$

in case equation (8) holds.

THEOREM V. *If $f_x'(x_0, s)$ exists, and if $p(s)$ is any integrable function, the function $H(x) = \int_E f(x, s)p(s)ds$ has a derivative at the point $x = x_0$, given by the expression*

$$(11) \qquad \frac{dH(x_0)}{dx} = \int_E f_x'(x_0, s)p(s)ds.$$

Let $\Delta H = [H(x_0 + h) - H(x_0)]$. Then, because of the Schwarz inequality for integrals, it follows that

$$\left|\frac{\Delta H}{h} - \int_E f_x'(x_0, s)p(s)ds\right| = \left|\int_E [f_x'(x_0, s) - q(x_0, h, s)]p(s)ds\right|$$
$$\leqq \sqrt{\int_E p^2(s)ds}\sqrt{\int_E [f_x'(x_0, s) - q(x_0, h, s)]^2 ds},$$

and this expression, because of (8), approaches zero with h. This establishes the equation (11).

Let us represent by $M(x)$ the square root of the quantity $\int_E f^2(x, s)ds$. Then we have the following theorem.

THEOREM VI. *Suppose that, at* $x = x_0$, $M(x_0) \neq 0$ *and that there exists* $f_x'(x_0, s)$. *Then, at* $x = x_0$, $M(x)$ *has the derivative*

$$\frac{dM(x_0)}{dx} = \frac{1}{M(x_0)}\int_E f_x'(x_0, s)f(x_0, s)ds. \tag{12}$$

We shall establish the existence of $dM^2(x_0)/dx$; from this result (12) will follow immediately. We obtain, by use of (7),

$$\frac{M^2(x_0 + h) - M^2(x_0)}{h} = \int_E \frac{f^2(x_0 + h, s) - f^2(x_0, s)}{h}ds \tag{13}$$
$$= h\int_E q^2(x_0, h, s)ds + 2\int_E q(x_0, h, s)f(x_0, s)ds.$$

As h approaches zero, the first term on the right in (13) approaches zero by (d), § 1. The second term approaches $2\int_E f_x'(x_0, s)f(x_0, s)ds$ because

$$\left|\int_E [q(x_0, h, s) - f_x'(x_0, s)]f(x_0, s)ds\right|^2$$
$$\leqq M^2(x_0)\int_E [q(x_0, h, s) - f_x'(x_0, s)]^2 ds,$$

which approaches zero with h. Consequently there exists

$$\frac{dM^2(x_0)}{dx} = 2\int_E f_x'(x_0, s)f(x_0, s)ds.$$

In the future suppose that E is a closed interval $c \leqq s \leqq d$. Consider an infinite system of integrable functions

$$I = [p_n(s); \quad n = 1, 2, \cdots],$$

which are unitary and orthogonal on E; that is,

$$(14)\quad \int_E p_i(s)p_j(s) = d_{ij} \qquad (d_{ii}=1,\ i=1,\ 2,\ \cdots;\ d_{ij}=0,\ i\neq j).$$

If $f(x, s)$ satisfies the conditions of this section of the paper, let

$$(15)\qquad z_i(x) = \int_E p_i(s)f(x, s)ds \qquad (i = 1, 2, \cdots).$$

For brevity we shall term the $z_i(x)$ the *Fourier coefficients* of $f(x, s)$ with respect to the system I and we shall call the vector $\xi(x) = [z_1(x), z_2(x), \cdots]$ the Fourier vector of $f(x, s)$. As an immediate consequence of Theorem V we may state that, if $f_x'(x_0, s)$ exists, the Fourier coefficients have derivatives at $x = x_0$ given by the expressions

$$(16)\qquad \frac{dz_i(x_0)}{dx} = \int_E p_i(s)f_x'(x_0, s)ds \qquad (i = 1, 2, \cdots).$$

If $f_x'(x, s)$ exists, let $\eta(x) = [y_1(x), y_2(x), \cdots]$ represent its Fourier vector with respect to the system I. It is seen from (16) that the y_i and z_i are related by the equation

$$y_i(x) = dz_i(x)/dx.$$

Let us suppose in the future that the system I is complete for the class of all integrable functions on E. That is, we assume that there does not exist any integrable function $k(s) \neq 0$ such that $\int_E k(s)p_i(s) = 0$ for $i = 1, 2, \cdots$. Then we shall establish the following theorem.

THEOREM VII. *For a value of x at which $f_x'(x, s)$ exists,*

$$(17)\qquad \lim_{h=0} \sum_{i=1}^{\infty} \left[\frac{z_i(x+h) - z_i(x)}{h} - y_i(x) \right]^2 = 0.$$

It has been proved by F. Riesz* that, if $g(s)$ and $h(s)$ are two integrable functions whose Fourier coefficients with respect to the system I are $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ respectively, then

$$\int_E g(s)h(s)ds = \sum_{i=1}^{\infty} a_i b_i.$$

From this Riesz formula, it is easily verified that the infinite

* Cf. Plancherel, loc. cit., p. 296.

sum in (17) is equal to

$$\int_E [q(x, h, s) - f_x'(x, s)]^2 ds,$$

so that Theorem VII is an immediate consequence of (8).

Before considering the next theorem, let us note some results which follow directly from the Riesz-Fischer* theorem concerning Fourier constants. If, for every x on (a, b), $f_x'(x, s)$ exists and is zero almost everywhere on E, then, because of (16), the Fourier coefficients of $f(x, s)$ are constants. Consequently, by the Riesz-Fischer theorem, an integrable function $g(s)$ exists such that for every x on (a, b), $f(x, s) = g(s)$. If $\xi(x) = [z_1(x), z_2(x), \cdots]$ is an infinite set of functions defined for x on (a, b) and is such that $\sum_{i=1}^{\infty} z_i^2(x)$ converges for all values of x, then there exists, for every x on (a, b), a unique function $w(x, s)$ with the following properties:

(*a*) $w(x, s)$ is integrable on E.

(*b*) $\xi(x)$ is the Fourier vector of $w(x, s)$ with respect to I. Let $\eta(x) = [y_1(x), y_2(x), \cdots]$ be a second set of functions with the same properties as $\xi(x)$, and let $u(x, s)$ be the function associated with $\eta(x)$ and having the characteristics corresponding to (*a*) and (*b*) above. Then, in regard to $\xi(x)$ and $\eta(x)$, we may state the following theorem.

THEOREM VIII. *If, for a certain x' on (a, b), $\xi(x')$ and $\eta(x')$ satisfy* (17), *the function $w(x, s)$ has a pseudo-derivative at $x = x'$ satisfying the equation $w_x'(x', s) = u(x', s)$.*

To establish the theorem consider the expression

$$\int_E \left[\frac{w(x' + h, s) - w(x', s)}{h} - u(x', s) \right]^2 ds. \tag{18}$$

By the Riesz formula, expression (18) is seen to equal the infinite sum entering in (17) with $x = x'$. Hence (18) approaches zero with h and, by definition, $w_x'(x', s) = u(x', s)$.

Let us now establish an analog of the mean value theorem, for functions possessing pseudo-derivatives.

THEOREM IX. *For every x on (a, b) suppose that $f_x'(x, s)$ exists and that*

$$\lim_{x'=x} \int_E [f_x'(x', s) - f_x'(x, s)]^2 ds = 0. \tag{19}$$

* Cf. Plancherel, loc. cit., p. 296.

Let us assume, also, that $f_x'(x, s)$ is integrable with respect to (x, s) in the rectangle $(a \leqq x \leqq b, c \leqq s \leqq d)$. Then it follows that, for all points (x, x') on (a, b),

$$(20)\quad f(x', s) - f(x, s) = (x' - x)\int_0^1 f_x'[x + u(x' - x), s]du.$$

In (20), it should be recalled that, for given values of (x, x'), the conventions of this paper permit that the equation should not hold for a set of points s with measure zero.

Let $\xi(x)$ be the Fourier vector of $f(x, s)$ relative to the complete system I. Then it is seen that the derivatives (16) exist and, moreover, that

$$(21)\quad \frac{dz_i(x)}{dx} - \frac{dz_i(x')}{dx} = \int_E [f_x'(x, s) - f_x'(x', s)]p_i(s)ds.$$

On applying the Schwarz inequality for integrals to the right side of (21), it is seen by (19) that the functions $dz_i(x)/dx$ are continuous for every value of x on (a, b). Therefore

$$(22)\quad z_i(x') - z_i(x) = (x' - x)\int_0^1 \frac{dz_i[x + u(x' - x)]}{dx}du$$
$$= (x' - x)\int_0^1 \left[\int_E p_i(s)f_x'[x + u(x' - x), s]ds\right]du,$$

where the last reduction was accomplished with the aid of (16). Let (x', x) be fixed and let us show that the left and the right sides of (20) have the same Fourier coefficients. From this result, as an immediate consequence of the Riesz-Fischer theorem, it will follow that equation (20) is satisfied.

Let $K(x, x', s)$ represent the right of (20). From the theory of Lebesgue integrals* it follows that $K(x, x', s)$ is defined for all values of s on E except possibly for a set E_1 of measure zero. If arbitrary values are given to $K(x, x', s)$ at points in E_1, K becomes an integrable function on the interval E. Let $(a_1, a_2, \cdots)$ be the Fourier coefficients of $K(x, x', s)$ with respect to I. Then we obtain

$$(23)\quad a_i = (x' - x)\int_E p_i(s)\left[\int_0^1 f_x'[x + u(x' - x), s]du\right]ds$$
$$= z_i(x') - z_i(x) \qquad (i = 1, 2, \cdots),$$

* Cf. de la Vallée Poussin, loc. cit., p. 53.

where the last reduction was made by interchanging integrations in (23) and by comparing the result with (22). The interchange was permissible* because of our present hypotheses. Since $[z_i(x') - z_i(x)]$ is the ith Fourier coefficient of the left side of (20), we have completed the proof of the theorem.

In a later paper the author will consider applications of the present results in the theory of functionals whose arguments are summable functions.

UNIVERSITY OF MINNESOTA,
December 15, 1920.

NOTE ON MINIMAL VARIETIES IN HYPERSPACE.

BY PROFESSOR C. L. E. MOORE.

(Read before the American Mathematical Society December 29, 1920.)

1. IT is known that a necessary and sufficient condition that a surface of two dimensions in hyperspace be minimal is the vanishing of the vector mean curvature.† It is the purpose of this note to show that mean curvature of a variety V_m in a space of n dimensions can be defined in the same way and that its vanishing is a necessary and sufficient condition that V_m be minimal. I shall use the absolute calculus, since one of its chief merits is the ease with which invariants can be written down. In fact the very form of an expression shows whether or not it is invariant. Enough vector analysis is used to simplify the form of the expressions.

The variety V_m can be written vectorially in the form

$$y = y(x_1, x_2, \cdots, x_m).$$

Then

$$ds^2 = dy \cdot dy = \sum_1^m a_{rs} dx_r dx_s.$$

If we write

$$y_r = \frac{\partial y}{\partial x_r},$$

* Cf. de la Vallée Poussin, loc. cit., p. 53.

† Wilson and Moore, "Differential geometry of two-dimensional surfaces in hyperspace," *Proceedings of the Amer. Acad.*, vol. 52 (1916).

we see at once that

$$a_{rs} = y_r \cdot y_s. \tag{1}$$

Hence we may write

$$dy \cdot dy = \Sigma y_r \cdot y_s dx_r dx_s.$$

Differentiating (1) covariantly with respect to the first fundamental form,

$$\varphi = \Sigma a_{rs} dx_r dx_s, \tag{2}$$

remembering that the first covariant derivative of the coefficients of the fundamental form is zero, we obtain

$$y_{rt} \cdot y_s + y_r \cdot y_{st} = 0,$$

and since this relation holds for any r, s, t, we may write

$$y_{ts} \cdot y_r + y_t \cdot y_{sr} = 0,$$

$$y_{rs} \cdot y_t + y_r \cdot y_{st} = 0.$$

The vector y is a function of $x_1, x_2, \cdots, x_m$. Since the second covariant derivative of a function is symmetric, we have

$$y_{rs} = y_{sr}.$$

From the last four equations we have

$$y_r \cdot y_{st} = 0, \quad r, s, t = 1, 2, \cdots m.$$

These equations show that y_{st} is perpendicular to y_r, and it is therefore perpendicular also to the tangent m-space of V_m.

Since the y_{rs} are vectors lying in the normal space, we can express them as linear functions of mutually perpendicular vectors in that space, which is of $n - m$ dimensions. Thus we find

$$y_{rs} = b_{1/rs} z_1 + b_{2/rs} z_2 + \cdots + b_{(n-m)/rs} z_{n-m}. \tag{3}$$

We have also the following relations:

$$z_i \cdot z_j = 0, \ (i \neq j); \quad z_i \cdot z_i = 1, \quad z_i \cdot y_s = 0. \tag{4}$$

Differentiating these covariantly, we have

$$z_{i/s} \cdot z_j + z_i \cdot z_{j/s} = 0; \quad z_{i/r} \cdot y_s + z_i \cdot y_{rs} = 0. \tag{5}$$

Then, multiplying (3) through by z_i, we have

$$z_i \cdot y_{rs} = b_{i/rs} = - z_{i/r} \cdot y_s.$$

2. *The Second Fundamental Forms.*—The coefficients $b_{i/rs}$ are generalizations of the coefficients of the second fundamental form of a surface in n-space.* They may be taken as the coefficients of $n - m$ fundamental forms which we can combine into a vector fundamental form

$$\psi = \Sigma y_{rs} dx_r dx_s. \tag{6}$$

We now consider the m systems of curves

$$\lambda_i^{(r)} = \frac{dx_r}{ds_i}, \qquad (i = 1, 2, \cdots m)$$

where ds_i is the element of arc along the curve λ_i. Multiplying by the proper factor, we can make the quantities λ_i satisfy the relation

$$\Sigma_r \lambda_i^{(r)} \lambda_{i/r} = 1.$$

We shall now assume that this relation is satisfied. Such systems are called unit systems. The λ's are called the coordinates of the congruence. If the congruences are orthogonal, the coordinates satisfy the relations

$$\Sigma \lambda_{i/r} \lambda_j^{(r)} = \epsilon_{ij},$$

$$\Sigma \lambda_{i/r} \lambda_i^{(s)} = \epsilon_{rs},$$

where $\epsilon_{kl} = 0$, if $k \neq l$, $\epsilon_{kk} = 1$. In terms of the coordinates, we can write the coefficients of the first fundamental form as follows:†

$$a_{rs} = \Sigma_i \lambda_{i/r} \lambda_{i/s}.$$

The coefficients of the second fundamental form can be written as follows:

$$y_{rs} = \Sigma_{i,\,j}\, \omega_{ij} \lambda_{i/r} \lambda_{j/s},$$

where ω_{ij} is a vector. The invariant vectors

$$\xi_i = \Sigma_r \lambda_i^{(r)} y_r = \Sigma_r \lambda_{i/r} y^{(r)} \tag{7}$$

are mutually orthogonal unit vectors tangent to the curves whose covariant coordinates are $\lambda_{i/r}$. Solving these equations for y_r and $y^{(r)}$ we find

$$y_r = \Sigma_i \xi_i \lambda_{i/r}, \quad y^{(r)} = \Sigma \xi_i \lambda_i^{(r)}.$$

* See Wilson and Moore, loc. cit., p. 308.

† Wilson and Moore, loc. cit., §13, p. 289, and §§26, 27, pp. 310–311.

We now introduce Ricci's coefficients of rotation*

$$\gamma_{hkl} = \Sigma_{r,s}\lambda_{h/rs}\lambda_k^{(r)}\lambda_l^{(s)}.$$

Solving these equations for $\lambda_{h/rs}$, we find

$$\lambda_{h/rs} = \Sigma_{i,j}\gamma_{hij}\lambda_{i/r}\lambda_{j/s}. \tag{8}$$

Substitution of this value in the covariant derivative of (7) gives

$$\xi_{i/r} = \Sigma_j\omega_{ij}\lambda_{j/r} + \Sigma_{j,l}\gamma_{ijl}\lambda_{l/r}\xi_j. \tag{9}$$

The condition that the congruence λ_i is a congruence of geodesics† is

$$\gamma_{jii} = 0.$$

3. *The Invariants* ω_{ii}. The curvature of a curve of the congruence λ_i which passes through a given point is obtained by differentiating ξ_i with respect to the arc length s_i. Thus we find

$$C_i = \frac{d\xi_i}{ds_i} = \Sigma_r \frac{\partial \xi_i}{\partial x_r}\cdot\frac{\partial x_r}{\partial s_i} = \Sigma\xi_{i/r}\lambda_i^{(r)},$$

which, by (9), reduces to

$$C_i \equiv \omega_{ii} - \Sigma_j\gamma_{jii}\xi_j.$$

The formula shows that the invariant vector ω_{ii} is the normal component of the curvature of the curve of the congruence in question. The tangential component of the curvature is

$$\gamma = \Sigma_j\gamma_{jii}\xi_j.$$

The invariant γ is then the geodesic curvature,‡ since the curvature of the projection on the tangential space is the projection of the curvature.§ If the congruence λ_i is geodesic, $\gamma_{jii} = 0$. Hence the curvature of a geodesic is directed along the normal and is equal to ω_{ii}.

An important invariant connected with any two differential forms is

$$-\Sigma a^{(rs)}y_{rs} = \Sigma a_{rs}y^{(rs)} = \Sigma_{ijkrs}\,\omega_{ij}\lambda_{i/r}\lambda_{i/s}\lambda_k^{(r)}\lambda_k^{(s)} = \Sigma\omega_{kk}.$$

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* Ricci and Levi-Civita, "Méthodes de calcul différentiel absolu," etc., *Math. Annalen*, vol. 54, p. 148.

† Ricci and Levi-Civita, loc. cit., p. 148.

‡ Ricci and Levi-Civita, loc. cit., p. 154.

§ Wilson and Moore, loc. cit., p. 320.

Hence the sum of the normal curvature vectors of m mutually orthogonal curves on the spread is constant. We shall define the mean curvature* h by the relation

$$mh = \Sigma\omega_{ii}.$$

4. *Minimal Varieties.*—Applying the preceding formulas, we can now easily derive the condition that the volume of a variety of m dimensions bounded by a variety of $m-1$ dimensions, chosen arbitrarily, shall be a minimum. The vector element of volume is

$$Pdx_1dx_2\cdots dx_m = \frac{\partial y}{\partial x_1}\times\frac{\partial y}{\partial x_2}\times\cdots\times\frac{\partial y}{\partial x_m}dx_1\,dx_2\cdots dx_m.$$

The condition for a minimum is

$$\delta\int(P\cdot P)^{\frac{1}{2}}dx_1dx_2\cdots dx_m = \int\frac{\delta P\cdot P}{(P\cdot P)^{\frac{1}{2}}}dV = \int\delta P\cdot MdV = 0,$$

where M is the unit tangent n-space

$$M = \frac{P}{(P\cdot P)^{\frac{1}{2}}}\cdot$$

Now

$$\delta P = \frac{\partial\delta y}{\partial x_1}\times\frac{\partial y}{\partial x_2}\times\cdots\times\frac{\partial y}{\partial x_m}+\frac{\partial y}{\partial x_1}\times\frac{\partial\delta y}{\partial x_2}\times\cdots\times\frac{\partial y}{\partial x_m}$$
$$+\cdots+\frac{\partial y}{\partial x_1}\times\frac{\partial y}{\partial x_2}\times\cdots\times\frac{\partial\delta y}{\partial x_m}.$$

The condition for a minimum then becomes the vanishing of n vector integrals. One of these integrals, after an integration by parts and the omission of the part which vanishes at both limits, becomes

$$\int\left(\frac{\partial\delta y}{\partial x_1}\times\frac{\partial y}{\partial x_2}\times\cdots\times\frac{\partial y}{\partial x_m}\right)\cdot MdV$$
$$= -\int\delta y\times\frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_2}\times\frac{\partial y}{\partial x_3}\times\cdots\times\frac{\partial y}{\partial x_m}\cdot M\right)dV.$$

* This is the direct generalization of mean curvature as used in the theory of surfaces. There is, however, a mean curvature definable in terms of the first fundamental form, and this is the one used by Einstein. The definition used in this note depends on the space in which V_m lies.

The complete condition then becomes

$$\int \delta y \times \left[\frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_2}\times \cdots \times \frac{\partial y}{\partial x_m}\cdot M\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_1}\times \frac{\partial y}{\partial x_3}\right.\right.$$

$$\left.\left.\times \cdots \times \frac{\partial y}{\partial x_m}\cdot M\right) + \cdots \pm \frac{\partial}{\partial x_m}\left(\frac{\partial y}{\partial x_1}\times \cdots \times \frac{\partial y}{\partial x_{m-1}}\cdot M\right)\right] = 0.$$

Since δy is arbitrary,

$$\frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_2}\times \frac{\partial y}{\partial x_3}\times \cdots \times \frac{\partial y}{\partial x_m}\cdot M\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_1}\times \frac{\partial y}{\partial x_3}\times \cdots\right.$$

$$\left.\times \frac{\partial y}{\partial x_m}\cdot M\right) + \cdots \pm \frac{\partial}{\partial x_m}\left(\frac{\partial y}{\partial x_1}\times \cdots \times \frac{\partial y}{\partial x_{m-1}}\cdot M\right) = 0,$$

which on multiplying out reduces to

$$\frac{\partial y}{\partial x_2}\times \frac{\partial y}{\partial x_3}\times \cdots \times \frac{\partial y}{\partial x_m}\cdot \frac{\partial M}{\partial x_1} - \frac{\partial y}{\partial x_1}\times \frac{\partial y}{\partial x_3}\times \cdots \times \frac{\partial y}{\partial x_m}\cdot \frac{\partial M}{\partial x_2} + \cdots = 0.$$

Now substituting

$$\frac{\partial y}{\partial x_r} = y_r = \Sigma_k \xi_k \lambda_{k/r},$$

$$M = \xi_1 \times \xi_2 \times \cdots \times \xi_m,$$

$$\frac{\partial M}{\partial x_i} = \frac{\partial \xi_1}{\partial x_i}\times \xi_2 \times \cdots \times \xi_m + \xi_1 \times \frac{\partial \xi_2}{\partial x_i}\times \xi_3 \times \cdots \times \xi_m + \cdots,$$

and using (9), the condition for a minimum becomes

$$(\Sigma_i \omega_{ii})\,|\lambda_{i/r}| = 0.$$

The determinant $|\lambda_{i/r}|$ is equal to $\sqrt{a}$, where a is the determinant of the first fundamental form, for from the expression $a_{rs} = \Sigma_i \lambda_{i/r}\lambda_{i/s}$ we have

$$a = |a_{rs}| = |\Sigma_i \lambda_{i/r}\lambda_{i/s}| = |\lambda_{i/r}|\cdot|\lambda_{i/s}| = |\lambda_{i/r}|^2.$$

Hence the condition for a minimum is $\Sigma\omega_{ii} = 0$.

The vanishing of the mean curvature vector is a necessary and sufficient condition that the variety be minimal. This is the same condition as that for a 2-surface in hyperspace.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
November 24, 1920.

NOTES ON ELECTRICAL THEORY.

BY PROFESSOR H. BATEMAN.

1. *The Production of Light.*—According to an idea that is generally associated with the names of Faraday and J. J. Thomson, energy is radiated from an electric charge in the form of light or electric waves when the lines of electric force issuing from the charge assume a wavy form. This idea has been made more definite by a study of the form of the lines of force in various types of electromagnetic fields. It is known now that waves on the lines of force may be produced either by an oscillation of the electric charge or by the continual emission of either electric or magnetic doublets. The emission of a single electric charge produces a kink in each line of force while the emission of a single magnetic charge produces a rotation of each line of force. When charges of opposite signs are successively emitted in the same direction so as to be equivalent to a doublet, the successive kinks are in opposite directions and are thus equivalent to a solitary wave. When a succession of magnetic poles of one sign are emitted in one direction, the lines of force rotate about this direction but they rotate with different angular velocities. The same is true when magnetic poles of opposite signs are fired out in opposite directions.

If a magnetic pole emitted in one direction is followed by a magnetic pole of the opposite sign so as to give a magnetic doublet, the lines of force rotate first in one direction and then in the other returning finally to their original directions. The result again is that a solitary wave runs along each line of force, but just as in the electrical case one line of force is unaffected.

The question now arises whether two types of light production really occur in nature. This is a question that has been bothering physicists for some time and opinion is divided. An argument in favor of the dual nature of light may be presented as follows:

2. *The Reaction of an Electric Charge upon a Field which Alters its Motion.*—It is usually assumed in electromagnetic theory that when an electric charge A is accelerated by an

external electromagnetic field E, the field E is modified indirectly only by the action of the electron's field on the sources of the field E. This reaction is, however, an after effect and it seems more in accordance with the principles of mechanics to suppose that the acceleration of A is accompanied by a direct local modification of the field E. Of course the acceleration of A produces a modification of A's field and it may be thought that this is the only local modification that occurs, but this modification spreads out in all directions while A's motion is changed only in one direction. At any rate the hypothesis that the field E is locally modified is worth considering; it may be that the disregard of this modification has been the cause of the present difficulties in the electromagnetic theory of radiation.

To get an idea of the nature of this local modification we shall adopt a definite hypothesis with regard to the structure of a line of electric force as viewed by an imaginary observer who is able to distinguish particles that are extremely small and exceedingly close together.

We shall suppose that a line of electric force issuing from a stationary positive charge P is built up of a series of doublets arranged as follows:

$$P+ \quad -+ \quad -+ \quad -+ \quad -+ \quad A \quad -+ \quad -+ \quad -+ \quad -+ \quad \rightarrow$$

These doublets may be supposed to be moving in the direction of the arrow with the velocity of light. The free charge at P is only temporarily at rest, for it may be supposed to secure a negatively charged partner from a doublet which arrives at P and move away with the velocity of light, leaving the positive constituent of the doublet at P until this charge in its turn secures a negative partner. The charge at P may be really moving all the time along a zig-zag path in the neighborhood of the point P, the mean free path being comparable in size with the so-called "diameter" of the positive charge. If the mass of the charge is proportional to the number of collisions with doublets per unit time it is easy to understand why the mass is inversely proportional to the "diameter" or mean free path of the free electric charge, especially if the free charge always moves with the velocity of light in describing its zig-zag path. If n denotes the number of collisions per second and m the mass of the positive charge, a lower limit to the value of n may, perhaps, be given by the equation $mc^2 = hn$,

where c denotes the velocity of light and h, Planck's constant. The value of n for the nucleus of a hydrogen atom is then of order 10^{23}. So also if an electron is similarly constructed the number n is of order* 10^{20}. Let us now suppose that an electric charge of the same magnitude as one of the constituents of a doublet meets the line of electric force at A. If the charge is positive it may be supposed to seize the negative constituent of the doublet on the right leaving the positive constituent free. The effect is the same as if the free positive charge had been moved to the right, *i.e.* as if it had been repelled by the free positive charge at P. As for the newly formed doublet, it may be supposed to fly away in some direction which depends, perhaps, on the direction of motion of the free charge which entered at A. It is possible that the colliding charges are broken up into fragments which are scattered in all directions. This is a possibility that ought to be considered but at present it is simpler to assume that the charges remain intact.

In the latter case the result of the encounter between the free electric charge A and the line of force is that the free charge is displaced along the line of force, one of the doublets belonging to the line of force is annihilated and another doublet is formed and emitted in some direction which is probably different from that of the line of force. Mathematically the annihilation of one doublet and the emission of another may be represented by the emission of two doublets one of which travels in the direction of the doublet that was annihilated and is of such a nature as to completely annul the effect of this doublet. Where the former doublet would have a positive charge the emitted doublet must have an equal negative charge.

Electromagnetic fields in which pairs of doublets are emitted from a moving point have already been studied.† They may be derived from the elementary type of field in which charges which are equal but opposite in sign are emitted in different directions.

When the electric charge which meets the line of force at A

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* If a free charge moves with the velocity of light, the mean length of the free path in the case of an electron may be of order 10^{-10}. This is about the order of magnitude of the radius of the ring electron. A free charge moving along a zig-zag path may be supposed to behave something like a ring electron.

† *Proc. London Math. Soc.*, (2), vol. 18 (1919), p. 95.

is negative, it may be supposed to seize the positive constituent of the doublet on the left, leaving the negative constituent behind. The result is equivalent to a displacement of the free charge towards the free positive charge at P just as if the negative charge were attracted by the positive charge. There is also a breaking up of a doublet belonging to the line of force and the emission of a new doublet in some direction. The displacement or acceleration of the free electric charge in the direction of the line of force may be regarded as an elementary process of which the equations of motion of an electric charge are a consequence. The free charge of an electron may be supposed to encounter at least 10^{10} lines of force from different charges in 10^{-10} seconds and the resultant acceleration may consequently be proportional to a mean value of the electric force in this interval of time. It is difficult to derive formally the equations of motion from our hypothesis regarding the structure of a line of force; the difficulty may arise from the fact that the true line of force of the familiar type of electromagnetic field is a limit of the discontinuous line of force which we have pictured in order to make the argument clear. The discontinuous structure may, however, be more nearly correct than the continuity with which we are so familiar. The idea of mass being proportional to a frequency of collision seems reasonable when compared with other physical laws such as the law of mass action in chemistry and Planck's relation between an energy quantum and frequency. The idea arose originally in connection with a theory of gravitation which has already been sketched elsewhere.*

3. *The Lines of Force of an Oscillating Electric Pole.*—It is known that a line of electric force of a moving electric pole can be considered as the locus of a series of light particles projected from the pole at successive instants.† If $\mathbf{v}$ denotes the velocity of the pole at time τ and the unit vector $\mathbf{s}$ indicates the direction of projection of the light particle emitted at this instant, $\mathbf{s}$ satisfies the differential equation

$$(c^2 - v^2)\mathbf{s}' = c\mathbf{s} \times \mathbf{v}' \times (c\mathbf{s} - \mathbf{v}),$$

where primes denote differentiations with respect to τ. When

* *Messenger of Mathematics*, vol. 48, Aug., 1918, p. 74.

† Leigh Page, *Amer. Journ. of Sci.*, vol. 38, Aug., 1914, p. 169; H. Bateman, this BULLETIN, March, 1915.

the motion is rectilinear and along the axis of z the above equation is easily solved. If (l, m, n) are the three components of $\mathbf{s}$, we have*

$$l = \frac{2\beta\sqrt{c^2 - v^2}\cos\alpha}{c + v + \beta^2(c - v)}, \qquad m = \frac{2\beta\sqrt{c^2 - v^2}\sin\alpha}{c + v + \beta^2(c - v)},$$

$$n = \frac{c + v - \beta^2(c - v)}{c + v + \beta^2(c - v)},$$

where α and β are constants for each line of force.

It is easy to see that n increases with v and is stationary when v is stationary. Let us consider the case of a simple periodic oscillation about the origin in which $v = ap\cos p\tau$. The extreme values of n are then given by $v = \pm ap$ and occur when the electric pole is at the origin. Let P be the position at time t of a light particle shot out in one extreme direction; then remembering that l', m', and n' are zero when the electric pole is at the origin, we see that as a point moves from P along the figures of the line of force at time t, the increments of the coordinates are

$$dx = -\,cld\tau, \qquad dy = -\,cmd\tau, \qquad dz = (v - cn)d\tau,$$

where $v = \pm ap$.

The direction of the tangent at P is found to be the same whether we take the upper or the lower sign in the expression for v. Let us draw a line OQ parallel to the tangent and study the relation of the line of force to this radial line OQ. The line of force evidently has a wavy form and P is the crest of one of the waves. The distance of P from the radius OQ is easily found to be

$$\frac{2\beta uc(t - \tau)}{[c^2(1 + \beta^2)^2 - u^2(1 - \beta^2)^2]^{1/2}},$$

where $u = \pm ap$. As we move from crest to crest going away from O, the height of a crest above OQ increases very slowly if u is small. Thus if we are considering the flight of a light particle, the height of a crest increases approximately at a rate

$$\frac{2\beta u}{1 + \beta^2},$$

* F. D. Murnaghan, *Amer. Journ. of Math.*, April, 1917; Johns Hopkins Circular, July, 1915.

which is equal to u when $\beta = \pm 1$ and is less than u for other values of β. When $\beta = 0$ the line of force is straight.

If the intensity of the ordinary light emitted from an oscillating electric pole depends on the square of the acceleration ap^2, then for a given intensity $u = ap$ is inversely proportional to p, so that for light of high frequency u may be very small indeed and the crests on a wavy line of force may be very close to the radial line OQ even at a great distance from the pole. It may be on this account that light of high frequency behaves much more nearly as if it were corpuscular in nature than light of low frequency.

The intensity of the light depends upon the slope of the line of force away from OQ rather than upon the distance of the crests from OQ. When v is very small in comparison with c the light vector at a great distance from O may be taken to be equal to $e\boldsymbol{s}'/cr$, where e is the electric charge associated with the pole and r is the distance from the pole.

4. *Lines of Electric Force in Some Other Types of Electromagnetic Fields.*—If we suppose that an emission of light resulting from an acceleration of an electron is accompanied by an emission of doublets owing to a local modification of the field which accelerates the electron, there is some uncertainty as to the way in which we should represent mathematically the resulting electromagnetic field when the light is of the frequency of visible light or even X-rays, for a large number of doublets may be emitted in a single period. It seems best at first to simplify matters by considering first fields in which doublets or charges are emitted in all directions and then some fields in which doublets and charges are emitted only in a few particular directions.

Let us first of all consider the field $(\boldsymbol{E}, \boldsymbol{H})$ given by

$$\boldsymbol{H} + i\boldsymbol{E} = \nabla\sigma \times \nabla\tau + \frac{i}{c}\left[\frac{\partial\tau}{\partial t}\nabla\sigma - \frac{\partial\sigma}{\partial t}\nabla\tau\right],$$

where $r = c(t - \tau)$ is the distance of a point P whose coordinates are (x, y, z) from a moving point Q whose coordinates at time τ are (ξ, η, ζ) and whose velocity is $\boldsymbol{v}$. The quantity σ is given by the equation

$$\sigma = \frac{(\boldsymbol{l}\cdot\boldsymbol{r}) + k}{(\boldsymbol{v}\cdot\boldsymbol{r}) - cr},$$

where $\boldsymbol{l}$ is a complex vector which is a function of τ and k is a function of τ. The vector QP is denoted by $\boldsymbol{r}$.

In this type of field both electric and magnetic charges are emitted in all directions from the moving pole Q and the electric charge associated with the pole may vary. When $\boldsymbol{l} = e\boldsymbol{v}'$ and $k = e(c^2 - v^2)$, we obtain the field of a moving electric pole from which no charges are emitted. If $k = e(c^2 - v^2)$ and the vector function $\boldsymbol{l}$ changes sign periodically or represents a rotating vector the electric and magnetic charges which are emitted may be regarded as forming doublets.

The lines of electric force may be considered as made up of light-particles projected from the pole Q when k is real. The differential equation satisfied by the unit vector $\boldsymbol{s}$ which indicates the direction of projection at time τ is

$$k\boldsymbol{s}' = \boldsymbol{p} + (\boldsymbol{s} \times \boldsymbol{q}) - \boldsymbol{s}(\boldsymbol{s}\cdot\boldsymbol{p}),$$

where

$$\boldsymbol{p} + i\boldsymbol{q} = c\boldsymbol{l} + i(\boldsymbol{v} \times \boldsymbol{l})$$

and both $\boldsymbol{p}$ and $\boldsymbol{q}$ are real vectors.

We are interested in the case when a line of force returns periodically to its original form. Let us consider uniform circular motion and write $\boldsymbol{v} \equiv (v \cos \omega\tau, \quad v \sin \omega\tau, \quad 0)$, $\boldsymbol{l} \equiv (-\ e\omega v \sin \omega\tau, e\omega v \cos \omega\tau, ib)$; then

$$\boldsymbol{p} \equiv [-\ (b + ec\omega)v \sin \omega\tau,\ (b + ec\omega)v \cos \omega\tau,\ 0],$$

$$\boldsymbol{q} \equiv (0, 0, cb + e\omega v^2).$$

It is easy to see that if $\boldsymbol{p} \neq 0$ the differential equation for $\boldsymbol{s}$ possesses only a finite number of solutions of type

$$\boldsymbol{s} = \frac{\alpha + \beta \cos \omega\tau + \gamma \sin \omega\tau}{f + g \cos \omega\tau + h \sin \omega\tau}$$

and that there are consequently only a finite number of lines of force which return to their original form after a period of time $2\pi/\omega$. In the case when $\boldsymbol{p} = 0$ and $k = e(c^2 - v^2)$ the differential equation for $\boldsymbol{s}$ reduces simply to

$$\frac{d\boldsymbol{s}}{d\tau} = \omega(\boldsymbol{n} \times \boldsymbol{s}),$$

where $\boldsymbol{n}$ is a unit vector in the direction of the axis of z. All

the solutions of this equation are periodic with a period $2\pi/\omega$ and so all the lines of force return to their original form periodically. The field in this case is of the type considered by Leigh Page.*

As an example of a field in which charges are fired out in particular directions let us consider the following expressions for the components of $\mathbf{E}$ and $\mathbf{H}$:

$$E_x = \frac{ex}{r^3} + \frac{\lambda y}{x^2 + y^2} - \frac{\nu z x}{r(x^2 + y^2)},$$

$$E_y = \frac{ey}{r^3} - \frac{\lambda x}{x^2 + y^2} - \frac{\nu z y}{r(x^2 + y^2)}, \qquad E_z = \frac{ez}{r^3} + \frac{\nu}{r},$$

$$H_x = \frac{\lambda z x}{r(x^2 + y^2)} + \frac{\nu y}{x^2 + y^2}, \qquad H_y = \frac{\lambda z y}{r(x^2 + y^2)} \cdot \frac{\nu x}{x^2 + y^2},$$

$$H_z = -\frac{\lambda}{r},$$

where λ and ν are functions of $\tau = t - r/c$.

We may write $\mathbf{E} = r\,d\mathbf{s} - c\mathbf{s}\,d\tau$, where $\mathbf{s}$ is a unit vector with components (l, m, n) which at time τ is in the direction of the radius vector r, provided

$$\frac{dl}{d\tau} = \frac{c\nu}{e}\,\frac{ln}{l^2 + m^2} - \frac{c\lambda}{e}\,\frac{m}{l^2 + m^2},$$

$$\frac{dm}{d\tau} = \frac{c\nu}{e}\,\frac{mn}{l^2 + m^2} + \frac{c\lambda}{e}\,\frac{l}{l^2 + m^2},$$

$$\frac{dn}{d\tau} = -\frac{c\nu}{e}.$$

These equations give

$$\frac{d}{d\tau}(l + im) = \frac{c\nu}{e}\,\frac{n(l + im)}{1 - n^2} + \frac{ic\lambda}{e}\,\frac{l + im}{1 - n^2},$$

consequently when n has been expressed in terms of τ, $l + im$ may be found by a simple integration. It may be noticed that when $\nu = 0$ and λ is a constant the lines of force revolve around the axis of z at different rates. If θ denote the angle which a line of force makes with the axis of z and ϕ denote

* *Proc. Nat. Acad. of Sci.*, March, 1920, p. 115.

its angular velocity around this axis we have $\sin^2\theta\cdot\dot{\phi} = c/e$. If each line of force carried a unit mass at unit distance from the origin the angular momentum of the mass would thus be the same for all the lines of force.

An attempt made previously* to obtain the lines of force when any system of charges or doublets is emitted from a moving pole is vitiated by an unfortunate oversight. It appears that Z cannot be made equal to unity as was assumed. The differential equations to be solved are consequently of type

$$g(\sigma, \bar{\sigma})\frac{d\sigma}{d\tau} = f'(\bar{\sigma}, \tau), \qquad g(\sigma, \bar{\sigma})\frac{d\bar{\sigma}}{d\tau} = f'(\sigma, \tau),$$

where g is a function† whose form is independent of the emitted system and consequently independent of the form of f'.

CALIFORNIA INSTITUTE OF TECHNOLOGY,
October, 1920.

SHORTER NOTICES.

Table de Charactéristiques de Base 30030, *donnant en un seul coup d'oeil les facteurs premiers des nombres premiers avec* 30030 *et inférieurs à* 901,800,900. By ERNEST LEBON. Tome I, Premier Fascicule. Paris, Gauthier-Villars, 1920.

To one who has spent eight years of his life in making a factor-table for the first ten millions the plan to extend such a table to the limit 901,800,900 seems like a rather serious undertaking. If such a table were constructed according to the plan devised by Burckhardt and employed by Dase and Glaisher the number of pages would exceed one hundred thousand, and with five hundred pages to the volume would fill some two hundred volumes! If, as in the tables published by the Carnegie Institute, the multiples of 2, 3, 5 and 7 were omitted, and the pages somewhat enlarged to take care of the large divisors certain to appear, the number of pages would be

* *Proc. London Math. Soc.*, (2), vol. 18 (1919), p. 123, *Phil. Mag.*, vol. 34, Nov., 1917, p. 419. In the notation used in the last paper the equations should be

$$g(\alpha, \beta)\frac{d\alpha}{d\tau} = f(\beta, \tau), \qquad g(\alpha, \beta)\frac{d\beta}{d\tau} = f(\alpha, \tau).$$

† In the case of a stationary pole, $g = (1 + \sigma\bar{\sigma})^{-2}$.

42,943, and the work would appear in 86 immense volumes! M. Lebon's table omits multiples of 2, 3, 5, 7, 11, 13; but a table of smallest divisors omitting such multiples, and constructed according to the extremely condensed plan of the Carnegie tables would still run into some 34,594 pages, or 69 large volumes!

Furthermore, M. Lebon notes the inconvenience of tables which give only the smallest divisors, and in his table proposes to list all the divisors! Such a plan, followed out in factor-tables of the kind already published, would certainly multiply their bulk by two, and probably by five. This Table de Charactéristiques is to take the place of three or four hundred volumes of the most compactly arranged factor-tables yet invented! If the author were not listed on the title page as Agrégé de l'Université, Professeur honoraire de Mathématiques au Lycée Charlemagne, Lauréat de l'Académie Français et de l'Académie des Sciences; if he were not able to cite favorable mention of it in the important mathematical societies of France, Italy, Spain and America; if the persons presenting his work before the various learned societies were not men like Volterra, Appell, Rouche, Neuberg, Nielsen, Darboux; if he had not been granted medals and prizes in recognition of his work, one would be inclined to dismiss the undertaking as absurd. With such a list of illustrious men and associations behind it one must take the publication seriously.

Of the 56 pages of tables contained in the Premier Fascicule of Tome I the first 40 are devoted to numbers of the form $Bk + 1$, where $B = 30030$, and give the factors of all such numbers for values of k not greater than B. The table is not arranged with increasing values of k, however, but with increasing values of I where the number $Bk + 1 = I \cdot I'$. The factorization of both I and of I' is given, and the corresponding value of k is also listed. This table of 40 pages not only serves to give the factors of all numbers not larger than B, but is also used, as we shall see, in finding the factors of larger numbers.

The second table, called a "Table of Characteristics," gives the factorization of numbers of the form $Bk + 1$ for k less than B, arranged with increasing values of k. The Premier Fascicule carries this table up only as far as $k = 4680$. There are 14 pages in this table and when complete up to $k = B$ it will cover some 90 pages altogether, unless the increasing number and size of the factors necessitate wider spacing be-

tween the columns. This table, then, when complete, will serve to give the complete factorization of all numbers of the form $Bk + 1$ within the limits proposed for the table. A similar table, giving the factors of all numbers of the form $Bk + I$, where I is any one of the 5760 numbers less than, and prime to B, would complete the factor table, but as such an extension of the table would mean 518,400 pages, or over a thousand large volumes, M. Lebon has another plan to propose. He undertakes to throw any number of the form $B\kappa + I$ into the form $Bk + 1$. This is indeed not difficult. Multiply the number $n = B\kappa + I$ by I' where $I \cdot I' = Bk + 1$. An easy reduction gives $nI' = BK + 1$ where $K = \kappa I' + k$. If, now, K is less than B we can find the factorization of nI' in the table of 90 pages noted above, I' being obtainable for any I in the first little table of 40 pages. But since I' may have any value less than B, and κ also may have any value less than B it is clear that K may have any value less than $B \times B = 901,800,900$, "Avec le Tome II commencera la table des charactéristiques $K > 30029$." It is worth while to pause to examine the extent of this "Tome II."

It is not quite true that K takes all values less than this limiting number, because those which give primes may be omitted. The number of pages necessary for the whole table will be, however, approximately B times the number necessary for the table as far as $K = B$, that is to say 90 pages. A simple multiplication gives then, 2,702,700 as the approximate number of pages in "Tome II." Truly a stupendous volume! Equal to 5,405 volumes of 500 pages each!

The reviewer believes, from his own experience in such matters that one page a day for the computation alone would be exceedingly rapid work. Allowing 300 working days for the year, "Tome II" will be completed some 9,000 years from now! Those who are looking forward eagerly to the appearance of this volume will be pained to note that M. Lebon has received from the Academy a notice "qui provoque un arrêt dans les calculs":

"Le Conseil d'Administration, d'accord avec la Commission technique, a décidé, dans sa dernière séance, qu'il y avait lieu d'attendre, pour continuer ses subventions, que la partie déjà exécutée de votre travail fût imprimée."

D. N. LEHMER.

Newton. By GINO LORIA. Rome, A. F. Formíggini, 1920. 69 pp.

THIS little volume, written by one who seems never to rest in his literary labors, is number 52 of a series of biographies issued under the rather poetic title of Profili. The title is quite appropriate, for each number is a kind of side glance at the profile of the individual whose biography is rapidly sketched by some worthy and skilled literary artist.

What Professor Loria has done is to set forth in popular style the incidents in Newton's life that are more or less known to mathematicians but are not so familiar to the general cultured public. It is needless to say that he has not attempted to present material not already known, since this was not his problem. In this case his is the mission of a popularizer. He has apparently drawn, directly or indirectly, from Brewster's well-known work, as all other historians of mathematics have done for the last two generations.

The life of Newton is briefly told,—his rather unpromising boyhood, the unusual promise shown by him at Trinity College, Cambridge, his power of easily grasping the theories of his predecessors, his rise to fame, his discovery of the laws of gravitation and of the fluxional calculus, and his later contributions to mathematics in general. The writer calls attention to the fact that the respect due to Newton is partly a case of hero worship, not less marked than the mental attitude of devout pilgrims to the Holy Sepulchre when in the presence of a portion of the true cross. In this he gives us the view of one who sees the Anglo-Saxon civilization from without, and it is a fair question whether we who see it from within have not been guilty of unduly exalting the contributions and powers of the author of the Principia. The fact is that Newton is today a good deal of a mystery, and there is need for a mathematical scholar of judicial mind, of literary ability, and of sufficient leisure to give us a new biography of the discoverer of the fluxional calculus, and a bibliography that is worthy of the man. We have Brewster and De Morgan and various minor writers, but we need an authoritative life of Newton and a definitive edition of his works, with a list, approximately complete, of published Newtoniana in general. It is very strange that we have modern editions of the works of the great mathematicians of Germany, France, and Italy, and of British scholars like Cayley and Sylvester,

but that Newton seems so sacrosanct that we feel that we must consult his works only in a first or second edition of two centuries ago, or in the Opera quae exstant omnia which Horseley published in 1779–1785. As an example of our lack of information, we do not really know why it is that Newton left Cambridge and spent the last thirty years of his life in London. The usual reasons are familiar, but no one of them seems sufficient to account for this substantially complete break with Cambridge; and if one will seek to ascertain, for example, what Newton was doing in 1720, he will very likely find an absolute blank. Did the story of his serious mental failure have any foundation in fact? We do not know from any authoritative evidence. We can say that on such a year he contributed an article on a certain subject, that at such a time he was corresponding with one of the Bernoullis, that he was president of the Royal Society during such a period, and we can link these facts together in a fragmentary fashion; but some capable historian is needed to take the problem up, basing his solution upon trustworthy documentary evidence and giving to the world a biography that shall be worthy of such a genius.

Needless to say that Professor Loria has not, in sixty-nine pages, attempted anything of this kind; but when he speaks of the state of hero worship that exists in all our Anglo-Saxon minds, with respect to Newton, he does us a good service. It is not a question of Newton's greatness,—it is a question of the precise facts by the knowledge of which we can properly measure that greatness.

It must not be thought that Professor Loria is an iconoclast, —far from it. Voltaire, in one of his baser moods, wrote that he had once believed that Newton was made the master of the mint because of his great merit, but that he "avait une nièce assez aimable, nommée Madame Conduit; elle plut beaucoup au grand trésorier Halifax. Le calcul infinitésimal et le gravitation ne lui auraient servi de rien sans une jolie nièce." If Professor Loria had a biased mind, he might have repeated the story without comment, but he shows, as we all know, that the statement was false and was unworthy the better nature of the "old invalid of Ferney."

The famous priority controversy relating to Newton and Leibniz is considered briefly, but the facts are now fairly well known and the dispute no longer has its former interest.

The Profili now sell for three lira each in Italy,—about fifteen American cents, at the present rate of exchange. They are artistically printed and each is the work of a scholar. It seems strange that we have never, in this country, been able to support a series of this kind. The chief criticism of the work of this Italian press lies in the number of typographical errors that appear. On a single page (68), for example, we have the "Leiters of Sis Isaac Newton," "R. Benthey" (for Bentley), "and other autenthic documents," "Eeral of Macclesfield, "Côtes" (for Cotes), and "Comptes-Rendus," while on page 66 there are no less than nine errors of a similar nature.

DAVID EUGENE SMITH.

Mathematiker Anekdoten. Zweite, stark veränderte Auflage. Von W. AHRENS. Teubner, Leipzig and Berlin, 1920. 42 pp.

WHETHER the telling of an anecdote shall provoke the interest of a pleased smile or the different amusement which leads to a shrug of the shoulders depends intimately and delicately upon the mental associations which arise involuntarily when a story is related; and the latter in turn depend upon the varied elements, and even the most minute, which make up the daily life and experience and environment. Hence it has always been, and perhaps always will be, difficult for one people to appreciate the humor of another. It is therefore natural that a book of anecdotes, containing humorous ones among others, shall be addressed by an author principally to his own countrymen.

These stories related by Ahrens of mathematicians and things mathematical are evidently intended primarily for his own countrymen; hence it is fitting that far the greater space should be given to men and things that are German. One of the pleasing features of the booklet is the inclusion of fifteen or more excellent likenesses of mathematicians. The stories range in excellence from some of high quality to some which are not pleasing. We do not find much of value in the story of the boys who convinced a simple old man that in their use of logarithm tables they were mastering the house numbers of Europe. We are only mildly interested when we are told of L. Fuchs' surprise when a long computation in his lecture led to the result $0=0$, that he first painfully suspected

an error, but that he then regained his composure and said "'Null = Null' ist ja ein sehr schönes und richtiges Resultat." But the story of the youth of Gauss will please every one who enjoys the activity of genius. An effective impression of the progress of mathematical instruction is made by a brief account of mathematical instruction in the "good old times." The account of the self-taught Arago's successes when examined on one occasion by Louis Monge and on another by Legendre is inspiring; any one who does not know these stories will be repaid if he looks up the booklet for them alone.

R. D. CARMICHAEL.

Leçons sur les Fonctions automorphes. Par GEORGES GIRAUD. Gauthier-Villars, Paris, 1920. 126 pp. [Collection de Monographies de Emile Borel.]

IN a course of lectures at the College of France the author of this monograph has treated several aspects of the theory of automorphic functions. His central object was to present from a single point of view properties of several sorts of automorphic functions as investigated by many authors following the lead of Poincaré and Picard. The present volume contains an exposition of a part of these lectures.

The plan of treatment proposed places certain limitations upon the choice of material and its method of organization. No attempt is made to give a complete exposition of the theory of the fuchsian functions of Poincaré. Of the topics not treated the following may be mentioned: the theorems on the representation of the coordinates of algebraic curves; those on the integration of linear differential equations with algebraic coefficients and regular singular points; generalizations of these for functions of more than one variable; the theory of the so-called Kleinian functions of Poincaré. A treatment of certain of the omitted topics, it is said (page 4), will be given later in the form of a memoir or perhaps in the form of another volume similar to the present one.

The author's principal purpose of unification is realized through the use of a certain general class of groups (Γ) satisfying certain hypotheses (H) of a rather general character. In the first instance the groups are given, not explicitly but only implicitly through their possession of certain properties. The postulational basis of this treatment is laid at the beginning

of the first chapter (pages 8–11); it is too long for reproduction in the review. The whole treatment proceeds in intimate dependence upon this logical basis. Certain central results are first obtained in association with any group for which hypotheses H are satisfied.

Chapter II is devoted to a class of linear groups: for the case of one variable these became the fuchsian groups whose properties have been treated by Poincaré; for the case of two variables they become the hyperfuchsian groups of Picard; for the case of more than two variables they are groups which have been studied by Fubini. All the groups of the class are shown to satisfy hypotheses H of the first chapter. Further properties of this special class are developed. In Chapters III and IV there is a similar treatment of certain quadratic groups, and of certain groups formed from a set of several given groups. The treatment in these four chapters (pages 8–90) is general and abstract in character and is intimately dependent upon the basic postulates H. The final Chapter V (pages 91–123) is devoted primarily to the functions of Poincaré. On account of the special features of the more restricted theory, certain results become more precise than in the more general theory; and this fact is brought out by a derivation of the detailed results.

R. D. CARMICHAEL.

Archimedes. By SIR THOMAS LITTLE HEATH. Society for the Promotion of Christian Knowledge, London, 1920. vi + 60 pp.

AFTER becoming familiar with the larger works which have made Sir Thomas Heath so widely known, the reader who takes up the little work under review will do so with a feeling of surprise. The academic world has come to expect from his pen only such extended treatises as he has written upon Apollonius of Perga, Diophantus, Aristarchus, Euclid, and Archimedes,—treatises filled with erudition and written in that classical style of which he is a master. If the reader is a man of the cloister, the surprise will be unpleasant; if he is a man among men, it will be the opposite. Since the spirit of the time makes scholars more and more men of the world, the balance of judgment is certain to be in favor—shall we say of the appellant or the defendant?

What Sir Thomas Heath has done is to give a brief and

popular summary of the results which he has set forth in his well known treatise on the great Syracusan, and in his pamphlet on the recently discovered manuscript (1906) on the method of treating mechanical problems. He has done this by means of seven brief and easily read chapters as follows: I. Archimedes; II. Greek Geometry to Archimedes (substantially the subject of the first volume of his forthcoming history of Greek mathematics); III. The Works of Archimedes; IV. Geometry in Archimedes; V. The Sandreckoner; VI. Mechanics; and VII. Hydrostatics. To this summary he has added a brief bibliography and a chronological table.

The story is told in a style that will easily appeal to the popular taste,—that is, to a taste that has not been trained to enjoy the more severe intellectual food served at the tables of erudition. The opening chapter, on the life of Archimedes, will fascinate any youthful reader and will be read with pleasure by those of more mature years. The chapter on Greek geometry requires a little knowledge of algebra and geometry, but can easily be read by any one with a high school education. The third chapter sets forth briefly the nature of the extant works of Archimedes and gives a list of those which are now known only by title. The chapter on geometry in Archimedes reaches a little further into the domain of mathematics, but will give the college freshman, in brief space, the essential information which he needs with respect to the contributions of the Greeks to the calculus. The sandreckoner is within the easy reach of the high school student, but the chapters on mechanics and hydrostatics will be found, in view of the denatured courses in physics in our American high schools, beyond the average pupil.

The frontispiece is from David Gregory's edition of Euclid (Oxford, 1703), already used in the author's work on Diophantus, but for some reason (perhaps the size of the page) there is no indication of the interesting source.

As a piece of technical bookmaking the work is, of course, not in the same class with the other publications of the author. The products of but few publishing houses can compare with those of the Cambridge and Oxford presses. Moreover, the book is published for popular use, and expense is a weighty consideration. The Society for the Promotion of Christian Knowledge has done and is doing a praise-worthy work in placing such works within the reach of every purse. It is a matter of great regret that we have not, in this country, a simi-

lar foundation which enables us to publish a series of works of this nature at a nominal price, as is done in several European countries. No better field than this is today open in this country for establishing a relatively small foundation which should seek to satisfy a hunger for good reading. With the work of this British society in mind, one can readily excuse the lack of an index, and the poor paper which war conditions have imposed.

DAVID EUGENE SMITH.

The Theory of the Imaginary in Geometry together with the Trigonometry of the Imaginary. By J. L. S. HATTON, principal and professor of mathematics, East London College. Cambridge, England, University Press, 1920. 216 pp. and 96 figures.

THE word theory in the above title is to be understood in a very non-technical sense. Indeed, apart from the idea of the invariant elements of an elliptic involution on a straight line, no theory is found at all. The purpose of the book is rather to furnish a certain graphical representation of imaginaries under a number of conventions more or less well known. Three concepts run through the work: first, an incompletely defined idea of the nature of an imaginary; second, the analogy with the geometry of reals; third, the use of coordinate methods, assuming the algebra of imaginaries.

Given a real point O and a real constant k, an imaginary point P is defined by the equation $OP^2 = -k^2$. The two imaginary points P and P' are the double points of an involution having O for center, and ik for parameter. The algebra of imaginaries is now assumed, and a geometry of imaginary distances on a straight line is built upon it. The reader is repeatedly reminded that in themselves there is no difference between real and imaginary points; that differences exist solely in their relations to other points. In the extension to two dimensions both x and ix are plotted on a horizontal line, while y and iy are plotted on a vertical line. Imaginary lines are dotted, and points having one or both coordinates imaginary are enclosed by parentheses, but otherwise the same figures are used for proofs, either by the methods of elementary geometry, or by coordinate methods.

In the algebra of segments it is shown that an imaginary distance $O'D'$ can be expressed in the form iOD, wherein OD is a real segment, or at most by OD times some number. Now

follows a long development of the extension of cross ratios, etc., to imaginaries. In fact every word of this is found implicitly in any treatment of the invariance of cross ratios under linear fractional transformation.

In Chapter II the conic with a real branch is introduced, beginning with involutions of conjugate points on lines having imaginary points on the conic. If the coefficients in the equation of a circle are real, the usual graph of $x^2 + y^2 = a^2$ for real x and real y is followed by replacing y by iy, then proceeding as before. The former locus is called the (1, 1) branch, and the latter the (1, i) branch of the circle. Similarly, it has a (i, 1) branch, and another, (i, i), but the latter has no graph. This idea is applied in all detail to ellipses, hyperbolas, and parabolas; in the case of the central conics it is also followed by replacing rectangular coordinates by a pair of conjugate diameters. The ordinary theorems of poles and polars, and the theorems of Pascal, Brianchon, Desargues, Carnot are shown to apply. Indeed, after having established the applicability of cross ratios in the earlier chapters, all these proofs can be applied in the same manner as to reals, without changing a word.

Imaginary angles are brought in in Chapter III. The reasoning by analogy has no meaning when applied to minimal lines. It is not shown why the results should be definite and unique in all other cases, but the statements on pages 71 and 73 are restricted by an undefined "in general." On page 73 it is tacitly assumed that the angle between the two bisectors of an imaginary angle is a right angle, and it is stated as a theorem later on that the sum of all the imaginary angles on one side of a straight line, measured at a point on the line, is two right angles. An imaginary line may rotate about a real line from a position of coincidence to one of perpendicularity with the real line. "A right angle may be divided into a series of imaginary angles." On page 70 we are told that "the point at infinity on the line is of the same nature as the base point. It can be regarded as real or imaginary." On page 91 we find: "the sine of θi increases from 0 to $i\infty$." After some skirmishing, trigonometry of imaginary angles is treated by the usual exponential formulas.

A curious mixture of premises introduces the critical lines, that is, those passing through the circle points. It is by no means clear that these are the only exceptions to be made in the application of the preceding theory.

The most satisfactory part of the entire treatise is that founded on the analytic basis. Here the reason for the exceptional nature of the critical lines becomes at once evident. The book could have been materially improved if the direct contradiction met in attempting to measure either distances or angles concerned with these lines had been pointed out.

Notwithstanding the preceding remarks, the book under review is not without merit. It is plainly and consistently written, the development frequently involves considerable skill, the results include a large number that are distinctly worth while, and the exercises furnish mental gymnastic material for many a lesson. The reviewer must contend, however, that it does not furnish a theory of imaginaries.

VIRGIL SNYDER.

COMMENT ON A PREVIOUS REVIEW.

In Professor Wilson's review of Eddington's *Space, Time and Gravitation,* which appeared in a recent number of this BULLETIN, he expresses his doubt as to the accuracy of Eddington's statement that Riemann never dreampt of a physical application of his analysis. That this doubt is amply justified and that Eddington's statement shows a lack of acquaintance with certain portions of the historical background of the general relativity theory, may be readily shown by a brief quotation from Riemann's famous Habilitationsschrift *Uber die Hypothesen welche dcr Geometrie zu Grunde liegen.* (We make use of Clifford's translation.)

"Either, therefore, the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it. The answers to these questions can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain. . . . This leads us into the domain of another science, of physics, into which the object of this work does not allow us to go to-day."

This statement (made in 1854) foreshadows the work of Einstein, and thus furnishes further evidence, if further evidence be deemed necessary, of the remarkable insight into mathematical and physical questions possessed by Riemann.

CHARLES N. MOORE.

NOTES.

At the Chicago meeting of the American Association for the Advancement of Science, Professor E. H. Moore was elected president for the next two years. Dr. R. S. Woodward, ex-president of the Carnegie Foundation and of the American Mathematical Society, was reelected treasurer, and Dr. B. E. Livingston permanent secretary, each for a term of four years. Professor Oswald Veblen was elected chairman of Section A (mathematics), and a vice-president of the Association. The winter meeting of 1921 will be held at the University of Toronto.

At the sixth annual meeting of the Mathematical Association of America, held at Chicago, December 28–29, the following officers were elected: president, Professor G. A. Miller; vice-presidents, Professors R. C. Archibald and R. D. Carmichael; members of the Board of Trustees, to serve until January, 1924, Professors A. A. Bennett, Florian Cajori, H. L. Rietz, and D. E. Smith. Professor C. F. Gummer was chosen to fill the vacancy on the Board caused by the election of Professor Carmichael as vice-president.

On the occasion of the mathematical congress held in Strasbourg in the summer of 1920, the following mathematicians were elected honorary members of the Society of Sciences, Agriculture, and Arts of the Lower Rhine: Professor L. Crelier, of the University of Bern; Professor L. E. Dickson, of the University of Chicago; Professor G. Koenigs, of the Sorbonne; Professor J. Larmor, of Cambridge University; Professor N. E. Nörlund, of the University of Lund; Professor E. Picard, of the Sorbonne; Professor C. de la Vallée Poussin, of the University of Louvain; Professor V. Volterra, of the University of Rome.

Announcement has just been made that Professor Dunham Jackson will present the principal paper of a symposium on the general theory of approximation by polynomials and trigonometric sums on Friday afternoon, March 25th, at the meeting of the American Mathematical Society at the University of Chicago. The meeting of the Society will extend over Saturday, March 26th.

On account of the meeting of the Central Association of Teachers of Mathematics and Science in St. Louis during Thanksgiving week of this year, the officers of the Southwestern Section of the American Mathematical Society and the officers of the Missouri Section of the Mathematical Association of America have decided to hold the meetings of these organizations also in St. Louis, at Washington University, instead of at the University of Missouri, as had been announced previously.

The opening (January) number of volume 22 of the TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY contains the following papers: *Arithmetical paraphrases*, by E. T. Bell; *The construction of algebraic correspondences between two algebraic curves*, by Virgil Snyder and F. R. Sharpe; *Concerning certain equicontinuous systems of curves*, by R. L. Moore; *Fundamental systems of formal modular seminvariants of the binary cubic*, by W. L. G. Williams; *A property of two* $(n+1)$*-gons inscribed in a norm-curve in n-space*, by H. S. White; *Recurrent geodesics on a surface of negative curvature*, by H. M. Morse; *On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function*, by J. L. Walsh; *On functions of closest approximation*, by Dunham Jackson.

The concluding (October) number of volume 42 of the AMERICAN JOURNAL OF MATHEMATICS contains the following papers: *Geometrical significance of isothermal conjugacy of a net of curves*, by E. J. Wilczynski; *Observations weighted according to order*, by P. J. Daniell; *Some determinant expansions*, by L. H. Rice; *A general implicit function theorem with an application to problems of relative minima*, by K. W. Lamson; *On the Laplace-Poisson mixed equation*, by R. F. Borden; *Characteristic subgroups of an abelian prime power group*, by G. A. Miller.

The December number (vol. 22, no. 2) of the ANNALS OF MATHEMATICS contains the following papers: *The mean of a functional of arbitrary elements*, by Norbert Wiener; *On certain determinants associated with transformations employed in thermodynamics*, by J. E. Trevor; *The permanent gravitational field in the Einstein theory*, by L. P. Eisenhart; *On the structure of finite continuous groups with a finite number of exceptional infinitesimal transformations*, by S. D. Zeldin; *Conformal*

mapping of a family of real conics upon another, by T. H. Gronwall; *On the location of the roots of the derivative of a polynomial*, by J. L. Walsh.

The January number (vol. 14, no. 1) of THE MATHEMATICS TEACHER announces the change in the management of that journal, which will be hereafter the official organ of the newly-formed National Council of Teachers of Mathematics. The new staff of editors is constituted as follows: J. R. Clark, editor-in-chief; E. R. Smith, assistant editor; and Alfred Davis, H. D. Gaylord, Marie Gugle, and J. W. Young. This staff will be assisted by an advisory board of twenty members under the chairmanship of Dean W. H. Metzler, formerly editor-in-chief of the same journal.

The March number of the AMERICAN MATHEMATICAL MONTHLY will contain a short statement concerning the scientific and editorial work of Professor F. N. Cole, who was, until the first of January of this year, the editor of this BULLETIN, and the secretary of the American Mathematical Society.

The October number of the BULLETIN OF THE NATIONAL RESEARCH COUNCIL (vol. 1, no. 5) consisted of a report on the quantum theory by Professor E. P. Adams. A few copies of this number are available for free distribution.

The Paris Academy of Sciences announces the award of the following prizes in pure and applied mathematics, in addition to those listed in the December BULLETIN (p. 139): the Grand prize of the mathematical sciences, to Ernest Esclangon, director of the Observatory at Strasbourg, for his memoir *New researches on quasi-periodic functions;* the Montyon prize for mechanics, to Stéphane Drzewiecki, for his work on the general theory of the helix, with special reference to aerial and marine propellers. The Academy announces the following problem for its Bordin prize, to be awarded in 1923: *To find all the cases in which the search for the surfaces admitting a given linear element leads to a partial differential equation of the second order that is integrable by the method of Darboux.* The Damoiseau prize was not awarded in 1920, no satisfactory memoir on the announced subject, *the figures of equilibrium of a rotating fluid mass subject to newtonian attraction* (see this BULLETIN, vol. 24, p. 316), having been received; the same problem is proposed again as the subject for 1923.

The Ackermann-Teubner memorial prize for 1920 was awarded to Professor Gustav Mie for his memoirs on the theory of matter which appeared in volumes 37, 39, and 40 of the ANNALEN DER PHYSIK.

The $5,000 prize offered through the SCIENTIFIC AMERICAN for a popular essay on the Einstein theories was awarded to Mr. L. Bolton, of London. The essay appears in the number of the SCIENTIFIC AMERICAN for February 5.

The Edison medal, awarded annually for work in electrical engineering by the American Institute of Electrical Engineers, will be presented this year to Professor M. I. Pupin, of Columbia University.

Announcement has been made at Brown University of the completion of the Nathaniel French Davis fund in honor of Professor Davis, now emeritus, who was for forty-one years a teacher of mathematics in that university. The fund amounts to ten thousand dollars, and the income is to supplement the regular library appropriations in purchasing mathematical books and periodicals for the mathematical seminary of Brown University.

A series of four lectures on *Four great geometers* (Archimedes, Galileo, Newton, and Maxwell) was given in November, 1920, by Mr. W. D. Eggar at Gresham College, London.

Professor Paul Painlevé, of the University of Paris, has returned from an educational mission to China. He has succeeded in arranging for an institute of the higher Chinese studies to be subsidized by the Chinese government in Paris, and for a branch of the University of Paris to be created in one of the Chinese universities and supported in part by the French government. An interesting account of these arrangements is given in the number of SCIENCE for February 4.

Professor Emile Borel, of the University of Paris, has been appointed honorary director of the Ecole normale supérieure.

Dr. David Owen, of Birkbeck College, has been appointed head of the department of physics and mathematics at the Sir John Cass Technical Institute, Aldgate, England.

Dr. P. Schafheitlin has been admitted as privat-docent at the technical school at Berlin.

Associate professor S. L. Boothroyd, of the department of mathematics and astronomy of the University of Washington, is spending the year at the Lick Observatory. He has been appointed to succeed Professor O. M. Leland as professor of astronomy and geodesy at Cornell University in September 1921.

Professor H. C. Wilson, of the department of mathematics and astronomy at Carleton College, has been granted leave of absence for the current academic year, part of which he will spend in research at Mt. Wilson Observatory. His position will be filled by Professor E. A. Fath, of Beloit College.

Associate professor Florence P. Lewis, of Goucher College, Baltimore, has been promoted to a full professorship of mathematics.

Dr. F. E. Wood, of the University of Chicago, has been appointed assistant professor of mathematics at the Michigan Agricultural College.

Dr. W. L. Crum, of Yale University, has been granted leave of absence for the second half of the present academic year to fill a temporary vacancy at Williams College as assistant professor of mathematics.

Dr. Teresa Cohen, of Johns Hopkins University, has been appointed instructor in mathematics at Pennsylvania State College.

Dr. Mary G. Haseman has been appointed instructor in mathematics at the University of Illinois.

Professor R. H. Weber, of the University of Rostock, died in August, 1920, at the age of forty-six years.

Dr. J. E. Clark, professor of mathematics in the Sheffield Scientific School of Yale University from 1873 to 1901, died January 3, 1921, in his eighty-ninth year. Professor Clark has been a member of the American Mathematical Society since its foundation.

Professor William Rinck, of Calvin College, Grand Rapids, Michigan, was killed in an automobile accident November 11, 1920, at the age of forty-three years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAUDET (P. J. H.). Het limietbegrip. Rede uitgesproken bij de aanvaarding van het ambt van hoogleeraar aan de Technische Hoogeschool te Delft. Groningen, Noordhoff, 1919. 19 pp.

BROCARD (H.). See MALO (E.).

BUTLER (N. M.). See KLAPPER (P.).

CAREY (F. S.). Infinitesimal calculus. Re-issue. New York, Longmans, 1919. 10 + 352 pp. $4.25

GÉRARDIN (A.). See MALO (E.).

GRÜNBAUM (H.) und JACOBI (J.). Funktionenlehre und Elemente der Differential- und Integralrechnung. 4te Auflage. Leipzig, Teubner, 1920. 194 pp. M. 6.00

HARDY (G. H.). Some famous problems of the theory of numbers and in particular Waring's problem. An inaugural lecture delivered before the University of Oxford. Oxford, Clarendon Press, 1920. 36 pp. 8vo. Paper. 2s. 6d.

HEATH (T. L.). Pioneers of progress: Archimedes. ("Men of Science" Series.) London, Society for the Promotion of Christian Knowledge, 1920. 8vo. 2 + 58 pp. Cloth. 2s.

HUYGENS (C.). Œuvres complètes. Tome 14: Calcul des probabilités. Travaux de mathématiques pures 1655–1666. The Hague, Nijhoff, 1920. 5 + 557 pp. 4to.

JACOBI (J.). See GRÜNBAUM (H.).

KLAPPER (P.). College teaching. Studies in methods of teaching in the college. Edited by P. Klapper with an introduction by N. M. Butler. [Chapter on the teaching of mathematics by G. A. Miller.] Yonkers, N. Y., World Book Company, 1920. 16 + 583 pp.

KNOPP (K.). Funktionentheorie. 2ter Teil. 2te Auflage. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1920. 138 pp.

MALO (E.). Sur le décomposition des nombres en leurs facteurs. Résolution des équations numériques. Préfaces de A. Gérardin et de H. Brocard. Nancy, Sphinx-Œdipe, 1920. 128 + 48 pp.

MILLER (G. A.). See KLAPPER (P.).

OS (C. VAN). Iets over de ontwikkeling der meetkunde. Rede, uitgesproken bij de aanvaarding van het ambt van hoogleeraar aan de Technische Hoogeschool te Delft. Groningen, Noordhoff, 1919.

POLETTI (L.). Tavoli di numeri primi entro limiti diversi, e tavole affini. (Manuali Hoepli.) Milano, Hoepli, 1920. 16mo. 20 + 294 pp. L. 10.50

STUYVAERT (M.). Algèbre à deux dimensions. Gand, Van Rysselberghe et Rombaut, 1920. 8vo. 223 pp.

ZEUTHEN (G. H.). Sur l'origine de l'algèbre. København, Høst, 1919.

ZIEHEN (T.). Lehrbuch der Logik auf positivistischer Grundlage mit Berücksichtigung der Geschichte der Logik. Bonn, A. Marucs und E. Webers Verlag, 1920.

II. ELEMENTARY MATHEMATICS.

AMMERMAN (C.). See FORD (W. B.).

BALL (W. W. R.). An introduction to string figures. An amusement for everybody. Cambridge, W. Heffer and Sons, 1920. 8vo. 38 pp. Paper. 2s.

FORD (W. B.) and AMMERMAN (C.). Teachers' manual, second course in algebra. New York, Macmillan, 1920. 5 + 247 pp. $1.20

GIRNDT (M.). Raumlehre für Baugewerkschulen und verwandte bautechnische Schulen. 2ter Teil: Dreiecksberechnung und Körperlehre. 5te, neubearbeitete Auflage. Leipzig, Teubner, 1919.

HOLTON (E. E.). See PADDOCK (C. E.).

PADDOCK (C. E.) and HOLTON (E. E.). Vocational arithmetic. New York, Appleton, 1920. 10 + 232 pp.

RUDERT (G.). Die Grundlagen des funktionalen Denkens in ihrer Bedeutung für den ersten mathematischen Unterricht. Berlin, Salle, 1919. 106 pp. M. 3.50

THIJN (A. VAN). Leerboek der stereometrie. Vijfde druk. Groningen, J. B. Wolters, 1919.

III. APPLIED MATHEMATICS.

ALLIAUME (M.). Cours d'astronomie. Partie générale élémentaire. Louvain, Uystpruyst-Dieudonné, et Paris, Gauthier-Villars, 1920. 8vo. 14 + 368 pp.

BADER (H. G.). Grundlagen der Flugtechnik. Entwerfen und Berechnen von Flugzeugen. Leipzig, Teubner, 1920. 8 + 194 pp.

BAYLE (G.). Cours de résistance des matériaux appliquée aux machines. Paris, Librairie de l'Enseignement technique, 1920. 8vo. 478 pp. Fr. 36.00

BOUASSE (H.). Théorie de l'élasticité. Résistance des matériaux. Paris, Delagrave, 1920. 8vo. 562 pp.

BURGATTI (P.). Lezioni di meccanica razionale. 2a edizione. Bologna, Zanichelli, 1919. 8vo. 12 + 545 pp.

CARMICHAEL (R. D.). The theory of relativity. 2d edition. (Mathematical Monograph Series, No. 12.) New York, Wiley, 1920. 112 pp. $1.50

CREHORE (A. C.). The new physics. San Francisco, published by the *Journal of Electricity*, 1920. 128 pp. $2.00

DELAFONTAINE (F. L.). Abaques pour le calcul des dalles, poutres et poteaux en ciment armé. Paris et Liège, Béranger, 1920. 4to. 24 pp. + 18 abaques. Fr. 24.00

DISPER (P.). Theorie der Entwicklung der Kometen aus den Prinzipien der Gravitation. Montabour, W. Kalb, 1919.

EINSTEIN (A.). Relativity: the special and the general theory. A popular exposition. Authorized translation by R. W. Lawson. London, Methuen, and New York, Holt, 1920. 13 + 138 pp. 5s.

FONTVIOLANT (B. DE). Les méthodes modernes de la résistance des matériaux. 2e édition. Paris, Gauthier-Villars, 1920. 8vo. 104 pp. Fr. 9.00

FÜRTH (R.). Schwankungserscheinungen in der Physik. (Sammlung Vieweg, Nr. 48.) Braunschweig, Vieweg, 1920.

GIANFRANCESCHI (G.). La fisica dei corpuscoli. 2a edizione. Torino, Fratelli Bocca, 1920. 8vo. 264 pp.

GRÜBLER (M.). Getriebelehre. Eine Theorie des Zwanglaufes und der ebenen Mechanismen. Berlin, Springer, 1918.

HAREN (R.). Aufgabensammlung zur Festigkeitslehre mit Lösungen. 2te Auflage. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1919. 89 pp.

KRAITCHIK (M.). Les tables graphiques financières. Paris, Gauthier-Villars, 1920. 4to. 20 pp. Fr. 7.50

LAWSON (R. W.). See EINSTEIN (A.).

MÜLLER (J.). Nautik. 2te Auflage. (Aus Natur und Geisteswelt, Nr. 255.) Leipzig, Teubner, 1919. 116 pp. Geb.

D'OCAGNE (M.). Principes usuels de nomographie avec application à divers problèmes concernant l'artillerie et l'aviation. Paris, Gauthier-Villars, 1920. 8vo. 4 + 70 pp. Fr. 9.00

RATEAU (A.). Théorie des hélices propulsives marines et aériennes et des avions en vol rectiligne. Paris, Gauthier-Villars, 1920. 8vo. 6 + 160 pp. Fr. 20.00

SAINTIGNON (F. DE). Le mouvement ininterrompu des molecules dans les fluides et autres hypothèses. Luxembourg, Imprimerie de la Cour Victor Buck, 1917. 4to. 175 pp.

SCHAU (A.). Statik. 1ter und 2ter Teil. 2te Auflage. Leipzig, Teubner, 1919.

VOLQUARDTS (G.). Feldmessen und Nivellieren. 4te Auflage. Leipzig, Teubner, 1919.

THE TWENTY-SEVENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The twenty-seventh annual meeting of the Society was held at Columbia University on Tuesday and Wednesday, December 28–29, 1920, extending through two sessions on each day. The attendance included the following eighty-six members:

Professor J. W. Alexander, Professor R. C. Archibald, Professor R. A. Arms, Professor Clara L. Bacon, Dr. Charlotte C. Barnum, Professor W. J. Berry, Mr. William Betz, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor R. W. Burgess, Professor B. H. Camp, Professor W. B. Carver, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Dr. W. L. Crum, Professor C. H. Currier, Dr. Tobias Dantzig, Professor J. V. De Porte, Dr. Jesse Douglas, Professor C. A. Fischer, Professor W. B. Fite, Professor C. H. Forsyth, Professor Tomlinson Fort, Mr. Philip Franklin, Dr. T. C. Fry, Professor R. E. Gilman, Professor O. E. Glenn, Professor W. C. Graustein, Dr. T. H. Gronwall, Professor Olive C. Hazlett, Professor E. R. Hedrick, Dr. A. A. Himwich, Professor L. A. Howland, Professor W. A. Hurwitz, Mr. S. A. Joffe, Professor Edward Kasner, Dr. K. W. Lamson, Mr. Harry Langman, Professor Gillie A. Larew, Professor Florence P. Lewis, Professor P. H. Linehan, Professor Joseph Lipka, Mr. L. L. Locke, Professor W. R. Longley, Professor C. R. MacInnes, Professor H. F. MacNeish, Professor R. M. Mathews, Professor H. H. Mitchell, Professor Frank Morley, Dr. H. M. Morse, Professor G. W. Mullins, Professor G. D. Olds, Professor F. W. Owens, Dr. Helen B. Owens, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Dr. E. L. Post, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. G. M. Robison, Dr. J. E. Rowe, Professor F. H. Safford, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Clara E. Smith, Professor P. F. Smith, Professor W. M. Smith, Professor Elijah Swift, Dr. J. S. Taylor, Professor H. D. Thompson, Mr. H. S. Vandiver, Professor Oswald Veblen, Professor J. N. Vedder, Professor J. H. M. Wedderburn, Professor H. S. White, Professor E. E. Whitford, Dr. Norbert Wiener, Professor A. H. Wilson, Professor Ruth G. Wood, Dr. T. S. Yang, Professor J. W. Young, Dr. S. D. Zeldin.

President Frank Morley occupied the chair, relieved by Professors G. D. Birkhoff and R. G. D. Richardson, on Tuesday; Professors Richardson and H. S. White presided at the sessions on Wednesday. The Council announced the election of the following persons to membership in the Society: Professor L. M. Coffin, Coe College; Professor I. H. Fenn, Polytechnic Institute of Brooklyn; Dr. Ludwik Silberstein, Eastman Kodak Company; Dr. W. L. G. Williams, Cornell University. One hundred twenty-one applications for membership were received.

A report was received by the Council from Professor E. R. Hedrick, chairman of the committee on increase of membership and sales of publications, presenting one hundred ten applications for membership in the Society and sixty-six subscriptions to the TRANSACTIONS. The report was accepted, with the thanks of the Council.

Professor T. S. Fiske, as representative of the contributors to the Bôcher memorial fund, tendered the fund to the Society to be held in trust and the income to be employed for the advancement of mathematical science. The trust was accepted, and a committee appointed to consider the most appropriate use to which the income of the fund could be devoted.

A committee was also appointed to make the necessary arrangements for the meeting of the Society to be held at Wellesley College in the summer of 1921.

The total membership of the Society is now 770, including 87 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 517; the number of papers read was 215. The number of members attending at least one meeting during the year was 306. At the annual election 189 votes were cast.

The Treasurer's report shows a balance of $8,994.53, including the Life Membership Fund, which amounts to $7,518.87. Sales of the Society's publications during the year amounted to $2,067.74.

The Library now contains 5,862 volumes, excluding some 500 unbound dissertations.

The afternoon session on Tuesday was especially marked by the retiring presidential address of Professor Frank Morley, on *Pleasant questions and wonderful effects.* A dinner was held at the Faculty Club Tuesday evening, at which fifty members were present.

At the annual election, which closed on Wednesday morning, the following officers and other members of the Council were chosen:

President,	Professor G. A. BLISS.
Vice-Presidents,	Professor F. N. COLE,
	Professor DUNHAM JACKSON.
Secretary,	Professor R. G. D. RICHARDSON.
Treasurer,	Professor W. B. FITE.
Librarian,	Professor R. C. ARCHIBALD.

Committee of Publication,
Professor E. R. HEDRICK,
Professor W. A. HURWITZ,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1923,

Dr. T. H. GRONWALL,	Professor FLORENCE P. LEWIS,
Professor O. D. KELLOGG,	Professor A. D. PITCHER.

At the close of the morning session on Wednesday, Professor H. S. White in a short address tendered the thanks of the Society to Professor Cole for his distinguished services during his long term of office as secretary of the Society and editor of the BULLETIN. Professor White said in part:

"With the close of the year 1920, the resignation of Professor Frank Nelson Cole ends his long period of service as secretary of the American Mathematical Society. Since 1895 he has held that office, and since 1897 has been editor of the Society's BULLETIN, a periodical issued ten times a year, and has published all the Society's programs and announcements. From these onerous duties, and others, he now withdraws.

"With deep regret we acquiesce. But we desire to express to our retiring secretary our gratitude for his labors. He has dignified and amplified his office. With few precedents to guide him, he has met with energy and tact its problems. The meetings of the Society he has prepared and managed with skill and eminent success. Its membership he has aided greatly to enlarge, maintaining cordial and helpful relations with all. As editor he has made the BULLETIN increasingly interesting and useful, securing an expanding line of active contributors. He has maintained its high standards, while extending its range to keep pace with the augmenting activities of the Society.

"Surely fortunate is he who is able to sustain for twenty-

five years labors so varied and so strenuous; who can through steady, cumulative efforts develop and execute so fully his plans and policies. We extend to him our congratulations, in grateful recognition of his achievements; and we felicitate him now on his return to the engrossing problems of pure science."

The list of papers read at the meeting, together with abstracts of the papers, is as follows:

(1) Professor C. E. Wilder: *Einstein's four-dimensional space is not contained in a five-dimensional linear space.*

In 1880 Voss gave necessary and sufficient conditions that any curved manifold be contained in any higher-dimensional manifold. Using these conditions, Professor Wilder shows that any four-dimensional manifold that is contained in a five-dimensional linear space cannot satisfy the defining equations for an Einstein space.

(2) Dr. J. L. Walsh: *On the convergence of the Sturm-Liouville series.*

Dr. Walsh considers the differential system

$$u''(x) + [\rho^2 - g(x)]u(x) = 0 \qquad (0 \leqq x \leqq 1,\ u(0) = u(1) = 0), \tag{1}$$

a special case of which is the differential system

$$u''(x) + \rho^2 u(x) = 0 \qquad (0 \leqq x \leqq 1,\ u(0) = u(1) = 0), \tag{2}$$

whose normal solutions are the functions $\{\sin n\pi x\}$. If $f(x)$ is an arbitrary function integrable and with an integrable square, there is formed the series composed of the term-by-term difference of the formal expansions of $f(x)$ in terms of the normal solutions of (1) and of (2); it is proved that this series converges uniformly and absolutely to zero on the entire interval considered. The method of proof is entirely elementary.

(3) Miss Anna M. Mullikin: *Certain theorems concerning connected point sets.*

Miss Mullikin establishes the following theorems.

I. If in a plane S, m_1, m_2, m_3, $\cdots$ is a countable collection of closed, mutually exclusive point sets, no one of which disconnects S, and if m, the sum of these point sets, is closed, then m does not disconnect S.

II. If in a plane S, P and Q are two closed point sets with no point in common and H is a closed, bounded, connected set of points containing at least one point in common with P and at least one point in common with Q, then H contains a connected subset H_1, such that (1) H_1 contains no point in common with P or with Q, and (2) each of the point sets P and Q contains at least one limit point of H_1.

(4) Dr. A. R. Schweitzer: *On homogeneous functions as generators of an abstract field.*

Dr. Schweitzer points out that the algebra of logic, the theory of abstract groups, and the theory of abstract fields, as special disciplines on iterative compositions, so far as is known, have been first emphasized by him. In further development of the abstract field as an iterative theory, he considers the possibility of generating an abstract field by rational functions of m variables ($m \geqq 2$), (1) homogeneous and integral of degree $\geqq 2$, (2) homogeneous and rational of degree zero. In particular, he finds that the following functions may serve as undefined relations for an abstract field: $x^2 - yz$, $(x + y)\cdot z$, $(x \pm y)\cdot y$, and $x/y - 1$, the latter function being also quasi-transitive. In connection with the latter functions, it is worthy of note that the non-homogeneous functions $(1 \pm x)\cdot y$ generate a field under suitable postulational assumptions.

(5) Dr. A. R. Schweitzer: *The concept of an iterative compositional algebra.*

Dr. Schweitzer defines an iterative compositional algebra as a set E of elements $a_1, a_2, \cdots, b_1, b_2, \cdots$, etc., subject to the following assumptions. Let $\lambda_i(a_jb_j)$ ($i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$) be generating relations of a given (formal or material, proper or improper) pseudo-group PG with a set (which may be vacuous) of adjoined conditions R on the functions λ_i; then the author defines

$$\Lambda_i(AB) = [\lambda_i(a_1b_1)\cdots\lambda_i(a_nb_n)]$$

where A and B are n-ads in the sense defined by him in the American Journal of Mathematics (1909). Let the n-ads $\{A\}$ be elements of compositions $r_s[A_1, A_2, \cdots, A_p]$, $p \geqq 2$, $s = 1, 2, 3, \cdots$, where the r_s satisfy a given set of equations in iterative compositions in the sense previously defined by

the author. In particular, closure under the compositions r may exist. The relation of the compositions λ_i to r_s will be the subject of special inquiry; in general, the λ_i and r_s satisfy explicitly stated equations in iterative compositions of order $\leqq i + s$. More generally, for the pseudo-group PG, one might substitute any set of necessary properties of any postulationally defined calculus of iterative compositions. See, e.g., Schroeder, ARCHIV (2), vol. 5.

(6) Professor Joseph Lipka: *Transformations of trajectories on a surface.*

In a paper read at the October meeting of the Society, Professor Lipka proved five geometric properties which completely characterize the trajectories on a surface for any positional field of force. In the present paper, a study is made of the point transformations which convert systems of curves on a surface S possessing some or all of these properties into like systems on the transformed surface S_1. The general results are that systems of curves possessing the first property or the first two properties are invariant under an arbitrary point transformation, while all other systems are only invariant under a geodesic transformation. A similar study is made for the point transformations of "n"-systems—brachistochrones, catenaries, velocity curves, etc.—on a surface.

(7) Mr. Harry Langman: *Conformal transformations of period n and groups generated by them.*

Professor Kasner has studied the groups of transformations generated by conformal transformations of period 2.

Mr. Langman's paper is principally concerned with a generalization of a problem studied by Kasner. The generalized problem, following Kasner's notation, consists in finding whether two transformations $f(z)$ and $g(z)$ can always be found such that $f_m(z) = g_n(z) = z$ and $g[f(z)] = F(z)$ identically for some integral values of m and n. The subscripts here denote the iterated functions of corresponding order. In the more general problem no such numerical relation as that found by Kasner for the case $m = n = 2$ is found necessary (other than $K_1^r = 1$). The complete result states that if $K_1 \neq 1$, then $F(z)$ is always factorable; if $K_1 = 1$, the periods of the factor transformations, if they exist, must be equal; if $K_1 = 1$ and $K_2 \neq 0$, then $F(z)$ is always factorable into two trans-

formations of arbitrary period > 2; if $K_1 = 1$, $K_2 = K_3 = \cdots = K_r = 0$, and $K_{r+1} \neq 0$, then $F(z)$ can not be factored into transformations of order r, or any factor of r, but can always be factored into transformations of any other order > 2.

In the first part of the paper it is shown that if the transformation $F(z)$ is of period n, then it may be put uniquely into implicit form. The well known fact that $F(z)$ is conformally equivalent to a rotation about the origin is then readily deduced. The method also gives complete solutions of similar functional equations of the symbolic forms: $f^m = g$, where g is given and $g^n = 1$; $1 + f + f^2 + \cdots + f^{n-1} = 0$; etc. It is also shown that, corresponding to every function f, where $f^n = 1$, there are others, F, satisfying the last functional equation.

Kasner has considered other transformations of period 2, termed by him *conformal symmetries*, defined by $f(z_0)$, where z_0 is the conjugate of z. It is here shown that no other kind of transformation of this type exists, i.e. that all *reverse* conformal transformations are of period 2.

(8) Professor O. E. Glenn: *On a new treatment of theorems of finiteness. Second paper.* (Preliminary report.)

Professor Glenn's paper consists, in its present form, of two parts. A third part, dealing with certain amplifications, is to be added. The first part comprises a proof of the finiteness of binary formal modular concomitant systems (modulo p), which has been an outstanding unsolved problem since the publication of Dickson's first paper on modular invariants (1907) and of a paper by A. Hurwitz (1903). This proof is based upon the algorism on modular concomitants appertaining to domains mentioned in the present author's memoir in the TRANSACTIONS for 1920. The second part of the paper gives statements and solution of certain finiteness propositions in the realm of differential invariants. The present paper is, in regard to general method, a sequel to the one published under the same title in the TRANSACTIONS for 1919.

(9) Professor J. E. Rowe: *The efficiency of projectile and gun.*

The purpose of this investigation is to formulate mathematically a means by which the efficiencies of different pro-

jectiles fired in different guns may be compared, using as a criterion the energy wasted in propelling the projectile to the point of fall. It is possible also to obtain from this the economic efficiency, by taking into account the cost of the material used.

(10) Dr. S. D. Zeldin: *On the structure of finite continuous groups with one two-parameter subgroup.*

This paper considers groups with one two-parameter subgroup. By imposing certain conditions on groups meroedrically isomorphic with the given ones, the author shows how the structure can be simplified.

(11) Dr. S. D. Zeldin: *On the structure of finite continuous groups with a finite number of exceptional infinitesimal transformations.*

In a previous paper, presented to the Society in December, 1919, Dr. Zeldin discussed the structure of finite continuous groups having one exceptional infinitesimal transformation. The present paper deals with the structure of groups having any finite number of exceptional transformations.

(12) Mr. H. S. Vandiver: *On quadratic congruences and the factorization of integers.*

In this paper, Mr. Vandiver considers the problem of the practical determination of the integral values x in the congruence $x^2 \equiv a \pmod{m}$ where a and m are rational integers prime to each other, a positive. A method of trial and exclusion is given, based on the theory of the quadratic form $u^2 - av^2$, which supplements another method due to H. J. S. Smith (*Collected Papers,* vol. 1, p. 148), who used a definite in lieu of an indefinite quadratic form. A similar procedure is convenient in finding the factors of an integer n, if n can be expressed in the form $y^2 - kz^2$, where k is a small positive integer.

(13) Professor E. V. Huntington: *A mathematical theory of proportional representation.*

When N representatives are distributed among the several states, the *true quotas* are usually fractional, and are necessarily replaced by whole numbers. Various methods of minimizing the injustice involved in these replacements have been proposed (by Willcox, Hill, d'Hondt, etc.), but without satis-

factory mathematical analysis. Professor Huntington shows that in the case of two states the *logarithmic error*, log (x/y), where $x =$ the ratio of the assignments as they actually are, and $y =$ the ratio of the assignments as they ought to be, provides a complete criterion for the *best* apportionment. If there are more than two states, the following principle is proposed: If the injustice belonging to any pair of states can be reduced by a transfer of representatives within that pair, this improvement should be made. It is shown that in any given case there is just one apportionment which is incapable of such improvement, and may therefore be regarded as the *best*. It is also shown how this *best* apportionment may be very easily found, by the use of a *priority list* formed by multiplying the population of each state by the series of factors

$$\infty,\ 1/\sqrt{1\cdot 2},\ 1/\sqrt{2\cdot 3},\ 1/\sqrt{3\cdot 4},\ \cdots,$$

and arranging the results for all the states in order of magnitude.

(14) Dr. H. M. Morse: *Recurrent motions of the discontinuous type.*

In investigating the nature of the trajectories in a stable dynamical system for unlimited values of the variable which represents the time, Professor Birkhoff has shown that one is led to types of motions which he calls recurrent motions, and which he divides into discontinuous and continuous types. Simple examples of motions of the continuous type can be given. The present paper by Dr. Morse offers the first proof of the existence of recurrent motions of the discontinuous type for the case where the motion is represented by a curve with a continuously turning tangent. This proof is the outcome of a complete classification, from the point of view of analysis situs, of geodesics on surfaces of negative curvature. The paper will be published in the TRANSACTIONS.

(15) Professor Frank Morley: Presidential Address: *Pleasant questions and wonderful effects.*

President Morley's address will appear in full in the April number of the BULLETIN.

(16) Professor Edward Kasner: *Properties of orbits in the general theory of relativity.*

The orbits in the Einstein theory are the geodesics of the

four-dimensional manifold M_4 defined by $ds^2 = \Sigma g_{ik}dx_idx_k$. Professor Kasner considers an arbitrary map of M_4 upon a 4-flat (x, y, z, t). The properties of the ∞^6 curves mapping the geodesics are studied. These curves are then projected orthogonally upon a 3-flat (x, y, z). The ∞^1 orbits corresponding to a given world-point (x, y, z, t) and a given direction $(dx : dy : dz)$ are shown to have the property that the locus of the centers of curvature is the inverse of a conic. If in particular the ds^2 is of the *static* form, studied by Levi-Civita and others, so that the world splits up into ordinary time and space, then it is shown that the conic becomes a circle through the given point; this means that the Meusnier property (relating to the variations of the curvature and the osculating plane) holds. In the Newtonian dynamics of a positional field of force, the locus becomes merely a straight line, since the ∞^1 trajectories then have a common osculating plane. Some further properties obtained for the general case are too complicated for brief statement.

(17) Professor Edward Kasner: *The solar gravitational field in finite form.*

In the last of his five notes on the Einstein theory presented at the summer meeting, Professor Kasner showed that there are no four-dimensional manifolds obeying Einstein's equations $G_{ik} = 0$ which can be immersed in a 5-flat; but that for a 6-flat there exist an infinity of solutions, including in particular the solar field. He now examines this field, taken in the Schwarzschild form, in detail. The final result is

$$ds^2 = -dx^2 - dy^2 - dz^2 + dX^2 + dY^2 - dZ^2,$$

where

$$X = \sqrt{\frac{r-2m}{r}}\sin t, \qquad Y = \sqrt{\frac{r-2m}{r}}\cos t,$$

$$Z = \sqrt{m}\sqrt{\frac{2r^3+m}{r^3(r-2m)}}\,dr,$$

where m is the mass of the sun, and $r^2 = x^2 + y^2 + z^2$. We thus have in finite form a model of the solar field. The model is situated in a flat space of six dimensions; or, using the more exact terminology of Weyl, in an affine-euclidean space of $(2 + 4)$ dimensions.

(18) Dr. Norbert Wiener: *The average of an analytic functional.*

Dr. Wiener discusses in this paper the notion of the average value of a functional which can be expanded in a series of multiple integrals, and develops an average which, over a considerable range of cases, is identical with that treated in his other papers, but which will apply directly to functionals that are not bounded.

(19) Dr. Norbert Wiener: *The average of a functional.*

Dr. Wiener gives a definition of the average value of a functional which reduces to a special case of the type of Daniell integral already treated by him in his paper entitled *The mean of a function of arbitrary elements,* presented to this Society at its meeting in December, 1919. The average thus developed is in intimate relation with the theory of probabilities, for it is based on the assumption that the distribution of the values of $F(T_1 + T_2)$, where F ranges over all continuous functions such that $F(T_1)$ has a certain determinate value X_1, is what is known statistically as a normal distribution and is independent of T_1 and X_1.

(20) Dr. Norbert Wiener: *Further properties of the average of a functional.*

Dr. Wiener defines the average of the bounded continuous functional F as

$$A\{F\} = \lim \pi^{-n/2}\prod^{n}(x_k - x_{k-1})^{-1/2}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} F\{\{\Psi\}(x)\}$$

$$\times \exp\left\{-\frac{y_1^2}{x_1^2} - \frac{(y_2-y_1)^2}{x_2-x_1} - \cdots - \frac{(y_n-y_{n-1})^2}{x_n-x_{n-1}}\right\}dy_1\cdots dy_n,$$

where $x_0, \cdots, x_n$ is a set of numbers satisfying the conditions

$$x_0 = 0, \qquad x_n = 1, \qquad x_{k+1} > x_k \text{ for every } k,$$

and where

$$\{\Psi\}(x) \equiv \{\Psi(x_1, \cdots, x_n)(y_1, \cdots, y_n)\}(x)$$

is that function of x which for $x_k < x < x_{k+1}$ assumes the value

$$y_k + \frac{x - x_k}{x_{k+1} - x_k}(y_{k+1} - y_k) \qquad (y_0 = 0).$$

The operation *lim* refers to the limit taken as the x's fill the interval from 0 to 1 more and more closely.

Dr. Wiener shows that $A\{F\}$ exists for every bounded continuous F. He shows, moreover, that the notion of average defined here is capable of a Daniell extension which is identical with the notion of average defined in the preceding paper.

(21) Professor Gillie A. Larew: *The Hilbert integral and Mayer fields for the problem of Mayer in the calculus of variations.*

A. Mayer and Bolza have shown that it is possible to extend to the problem of Lagrange in the calculus of variations the Hilbert theory in which an invariant integral is utilized in the proof that a certain set of conditions is sufficient for the existence of an extremum. A similar extension of this theory to the problem of Mayer is the subject of Professor Larew's paper. An integral is constructed effective in establishing a Weierstrass theorem, and it appears that the imposing of conditions for this integral to be independent of the path is equivalent to placing on the field of extremals conditions analogous to those characterizing the *Mayer fields* well known to students of the Lagrange problem. The study is confined to a Mayer problem in non-parametric representation and with fixed end-points. It may be noted that Kneser, using other methods, has found results consistent with these for the same problem in parametric form.

(22) Professor R. M. Mathews: *Generalizations of the classical construction of the strophoid.*

In the classical construction of the strophoid the curve is defined as the locus of the intersection of a pencil of circles with a pencil of lines. The circles have their centers on a line l and all pass through a point O on l. Each circle corresponds to that line of the pencil (F) which passes through its center. O is the node, F is the singular focus, and l is parallel to the asymptote. Professor Mathews generalizes this construction in two ways: first, for a pencil of circles through the node with their centers on any line through that point; and second, for a pencil of circles through any two points whose tangents meet on the curve.

(23) Professor W. A. Hurwitz: *Some properties of methods of evaluation of divergent sequences.*

In case a linear transformation, $y_n = \Sigma_{k=1}^{n} a_{nk}x_k$, carries every convergent sequence (x_n) into a sequence (y_n) converging to the same value as (x_n), the transformation is said to be regular; necessary and sufficient conditions for regularity are furnished by the Silverman-Toeplitz theorem. In the present paper, Professor Hurwitz investigates the conditions under which a regular transformation of real elements maintains the property of regularity even in the case of the improper limits $\pm\infty$, and also the conditions under which it carries a divergent sequence into another which, if it is not convergent, at least has its limits of oscillation not further separated than in the original sequence. Application is made to the particular case of *analytically regular* transformations, as defined by Hurwitz and Silverman.

(24) Professor W. C. Graustein: *Parallel maps of surfaces.*

Two real surfaces, in continuous one-to-one point correspondence, with the directed normals parallel at corresponding points, are said to correspond by a parallel map; or more specifically, by a directly parallel or an inversely parallel map, according as corresponding directions of rotation about corresponding points are the same or opposite. In Professor Graustein's paper, parallel maps are further classified as hyperbolic, elliptic, or parabolic, after the manner of the classification of one-dimensional projective correspondences, and each parallel map is characterized by an invariant analogous to the invariant of such a correspondence. The classification of parallel maps thus obtained is instructive in the light it sheds on known theorems concerning parallel maps, and it proves to be of value in developing new results.

(25) Professor J. H. M. Wedderburn: *On the maximum value of a determinant.*

Professor Wedderburn gives a proof, due to J. Schur (MATHEMATISCHE ANNALEN, vol. 66 (1909), pp. 488–510), of Hadamard's theorem that the maximum value of the determinant of a matrix $A = (a_{rs})$ is $|A| \leqq \Sigma(|a_{rs}^2|/n)^{n/2}$. As this inequality was not the main purpose of Schur's paper, his exposition does not show how brief the proof really is. If $\bar{a}_{rs}$ denotes the conjugate imaginary of a_{rs} and $\bar{A} = (\bar{a}_{rs})$, then $\bar{A}'A$ is a positive Hermitian form whose roots, as is well known, are real and positive. The sum of these roots is $\Sigma_{rs}a_{rs}\bar{a}_{rs} = \Sigma|a_{rs}^2|$;

hence their product is less than or equal to $(\Sigma|a_{rs}^2|/n)^n$. This product is equal to the determinant of $A'\bar{A}$ which is equal to the absolute value of $|A|^2$, thus proving the required inequality.

(26) Professor J. H. M. Wedderburn: *On the automorphic transformation of a bilinear form.*

The problem of the automorphic transformation of a bilinear form is usually discussed on the basis of the Cayley-Hermite solution. This solution is rational but becomes indeterminate under certain conditions. Professor Wedderburn derives a very simple solution in transcendental form which includes the exceptional cases and reveals their analytical nature clearly.

(27) Professor J. W. Lasley, Jr.: *Some special cases of the flecnode transformation of ruled surfaces.*

The process of obtaining the flecnode surface of a ruled surface leads to the consideration of a suite of surfaces, called the flecnode suite. Questions arise as to the cases in which this suite terminates or returns into itself. It is the purpose of Professor Lasley's paper to study some of these questions. It is found that the suite terminates with its first transform if and only if the given ruled surface has a straight line directrix. In case the termination occurs with the second transform, both branches of the flecnode curve on the given ruled surface may be obtained without integration. The suite is of period two if and only if the flecnode curve meets every generator in two coincident points. The flecnode suite cannot be of period three, nor of period four. The conditions for termination and periodicity considered are expressed in terms of the invariants of the differential equations which define the given ruled surface.

(28) Professor R. G. D. Richardson: *The theory of relative maxima and minima of quadratic and hermitian forms and its application to a new foundation for the theory of bilinear forms. First paper: Equivalence of pairs of bilinear forms.*

The theory of maxima and minima, which plays an ever increasing rôle in analysis, suggests a simpler and direct method for the discussion of a pair of forms. In the case of quadratic or hermitian forms, one of which is definite, each zero of the λ-determinant can be interpreted as a maximum

of a certain isoperimetric problem and as a minimum of another. This point of view enables Professor Richardson in this simplest case to obtain a direct method for reducing the pair of forms to the normal type as follows. Denoting by $\lambda_1, \cdots, \lambda_n$ the zeros of the λ-determinant, and setting $\lambda = \lambda_1$ in the corresponding λ-matrix $||a_{ij} - \lambda b_{ij}||$, the rows are regarded as the coefficients of linear homogeneous equations and the solution used as the first row of a matrix. The other rows are obtained in the same way by setting in the matrix $\lambda = \lambda_2, \cdots, \lambda_n$. The elements of the columns also serve as coefficients of linear equations which determine a second matrix. In the case of quadratic forms, this is identical with the first. When the λ-matrix is multiplied by these two matrices it is reduced to one with terms in the main diagonal only.

By a generalization of this process suggested by the formal work in the problem of taking the relative extreme, any pair of non-singular bilinear forms is reduced to its normal type. When the multiplicity of any λ_i is greater than its index (the number of linearly independent solutions of the corresponding linear equations) some of the rows and columns of the multiplying matrices are solutions of non-homogeneous equations extended from the original homogeneous equations by a process resembling the taking of a derivative.

The equivalence of pairs of bilinear forms is then based on three simple notions: (1) the equivalence of polynomials in λ; (2) the equality of the indexes of λ_i; (3) the possibility of solution of certain non-homogeneous linear equations.

(29) Dr. J. S. Taylor: *The analytic geometry of complex variables with some applications to function theory.*

Dr. Taylor's paper lays the foundations for a development of the analytic geometry of complex variables and applies some of the results to a geometric interpretation of function theory. A pair of complex numbers X and Y is represented by a point in four-space. Several definitions of distance are discussed, a geometric meaning is given to a complex angle, and six new pseudo-trigonometric functions of a complex angle are defined. The surfaces representing equations of the first and second degree in two complex variables are investigated in detail. Finally, a comparatively simple geometric meaning is given to the derivative of a function of a single complex variable, to its differential, and to its integral.

(30) Professor C. H. Forsyth: *The value of a bond to be redeemed ultimately, both principal and interest, in equal installments.*

The formulas for computing the value of a bond to yield a specified rate of interest have been derived for all cases where the principal is to be redeemed in one sum or in equal installments. The formula has been derived for the case where both principal and dividend are to be redeemed in equal annual installments. Professor Forsyth derives the formula for the more general case where the principal and dividend are to be redeemed in equal installments, but where the first installment is to be repaid at the end of f years and the rest at intervals of t years.

(31) Professor C. H. Forsyth: *Valuation of bonds bought to realize a specified rate of interest, assuming the amortizations to accumulate at a savings-bank rate.*

Investors in bonds realize only too well the usual impossibility of obtaining the rate of interest corresponding to the price computed by the ordinary formulas simply because the amortizations are relatively small sums, and small sums can rarely demand as high rates of interest as can such large sums as the principals of the bonds. Professor Forsyth derives a formula for computing the price of the conventional bond to yield actually a specified rate of interest since the amortizations are assumed to accumulate at a practical rate of interest, such as that of a savings bank. The formula thus derived proves singularly convenient for the solution of the inverse problem, which is the determination of the rate of interest that a given price will yield. In fact, the solution of the inverse problem is almost as easy and quite as simple as that of the direct problem.

(32) Dr. Einar Hille: *Zeros of Legendre functions.*

In this paper Dr. Hille considers solutions of Legendre's differential equation

$$(1 - z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + a(a + 1)w = 0$$

with respect to the location of their zeros in the complex plane. The distribution of zeros along rays from $z = +1$ is investigated by means of two fundamental integral equalities which

are generalizations of well known formulas generally attributed to Green. It is proved that $P_a(z)$, the solution regular at $z = +1$, has its zeros in the interval $(-1, +1)$ when a is real, and in the interval $(+1, +\infty)$ when $a = -\frac{1}{2} + \mu i$. The number of zeros has been previously determined by Dr. Hille. When $a = \lambda + \mu i$ $(\lambda > -\frac{1}{2}, \mu > 0)$ it is proved that the zeros lie in a certain sector of the upper half of the plane. Finally, the solution $Q_a(z)$ which belongs to the exponent $1 + a$ at infinity is shown to have no zeros in the finite plane as long as $\lambda > -\frac{1}{2}$.

(33) Professor W. B. Carver: *Systems of linear inequalities.*

In a paper on *Systems of linear inequalities* (ANNALS OF MATHEMATICS, vol. 20, p. 191, March, 1919), Professor L. L. Dines gives a necessary and sufficient condition for the existence of solutions of such a system. His conditions are expressed in terms of the I-rank of the matrix, the definition of I-rank being somewhat analogous to that of rank.

In the present paper Professor Carver gives a necessary and sufficient condition that solutions of the system should *not* exist, the condition being expressed in a form similar to that ordinarily used to define linear dependence. Independence of a system and equivalence of two systems are then defined, and necessary and sufficient conditions found for them.

(34) Professor J. L. Coolidge: *Differential geometry of the complex plane.*

Professor Coolidge's paper is devoted to the study of systems of points in the complex plane, whose coordinates are analytic functions of two real variables, yet which do not lie on a real curve. Some attention is also paid to three-parameter systems.

(35) Professor C. L. E. Moore: *Note on minimal varieties in hyperspace.*

Professor Moore's paper appeared in the February number of this BULLETIN.

(36) Professor I. J. Schwatt: *Independent expressions for the Bernoulli numbers.*

Laplace, Stern, Kronecker, Shovelton and others have developed independent expressions for the Bernoulli numbers. Professor Schwatt has established methods which have

enabled him to find new independent expressions for these numbers. The methods are believed to be more direct and the results more simple than those obtained heretofore.

(37) Professor I. J. Schwatt: *Relations involving the numbers of Bernoulli and Euler.*

By the methods of the preceding paper, Professor Schwatt has developed new relations involving the numbers of Bernoulli and Euler.

(38) Professor I. J. Schwatt: *Independent expressions for Euler numbers.*

Very few independent expressions for the Euler numbers have been developed. The simplest among them is the one by Worpitzky (JOURNAL FÜR MATHEMATIK, vol. 94, p. 203). The expressions by Scherk (ibid., vol. 4, p. 299), and by Saalschütz (*Vorlesungen über die Bernoullischen Zahlen*, p. 100) are rather complicated. By means of operations with series Professor Schwatt has been enabled to devise methods and to obtain results which are believed to be quite simple.

(39) Professor I. J. Schwatt: *Independent expressions for the Euler numbers of higher order.*

Lucas (BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE, vol. 6, p. 53) has introduced the Euler number of higher order. Professor Schwatt has been able to find only two independent expressions representing these numbers, one by Ely (AMERICAN JOURNAL OF MATHEMATICS, vol. 5, p. 339) and the other by Shovelton (QUARTERLY JOURNAL OF MATHEMATICS, vol. 46, p. 220). By means of operations with series methods have been devised which lead to simple independent expressions of these numbers. Three independent expressions for T_{np}, the coefficient of $x^{2n+p}/(2n+p)!$ in the expansion of $\tan^p x$, are also given.

(40) Professor I. J. Schwatt: *Summation of a type of Fourier's series.*

The principles used in the direct summation of

$$\sum_{n=0}^{\infty}(-1)^n\frac{\sin(a+ng)}{b+nh}r^n,$$

when r may be 1, in which case $g \equiv 0 \pmod{2\pi}$, is first applied to $\sum_{n=0}^{\infty}(-1)^n[1/(b+nh)]$. The method given by Professor

Schwatt in this paper is more general in its application than the one which he has presented in an article in the NOUVELLES ANNALES DE MATHÉMATIQUES, (4), vol. 6, pp. 203–209 (May, 1916).

(41) Professor F. W. Owens: *On the projectivity assumption in projective geometry.*

Various forms of the projectivity assumption in projective geometry have been made by different writers. A number of these are given and shown to be equivalent in Veblen and Young's *Projective Geometry.* In this paper, Professor Owens derives the ordinary forms of statement from milder forms of the assumption.

(42) Professor R. W. Burgess: *On certain simple skew frequency curves.*

To facilitate the discussion of frequency distributions which do not conform to the normal probability curve, Pearson in 1895 obtained a series of skew frequency curves by integrating the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a + bx}{c + dx + ex^2}.$$

This equation is a generalization of the slope property of the normal curve

$$\frac{1}{y}\frac{dy}{dx} = -ax.$$

The aim of the present paper is to give, by considerations based on the elementary theory of probability, a new basis for some of Pearson's curves, and to propose modifications in his methods of fitting these curves to given data. Professor Burgess' point of view is that the individual items summarized by the frequency curve may be regarded as being deviations all in one direction from a fixed origin, and, in the discrete case, as being built up by successive increments, the probability of adding each increment being variable. Among the curves derived on this basis are those of the family $y = cx^n e^{-kx}$. These curves are one type of those introduced by Pearson, and are also cases of Knibbs' *flexible curve.*

In view of the fundamental importance of the area in work with frequency curves, Professor Burgess proposes to use, as the criterion for goodness of fit, the sum of the differences of

area from the origin to various values of x, between the actual frequency distribution and the trial curve. The trial curves for this method are obtained by the use of a table giving the percentage of area to the right of the centroid for important values of n. This method requires the construction of tables of areas of these curves, that is, of the integral

$$I(x, n) = \int_0^x x^{n-1}e^{-x}dx/\Gamma(n).$$

Such tables may be easily constructed for values of n which are integral multiples of one-half by the use of the difference equation

$$I(x, n+1) - I(x, n) = -y(x, n+1)$$

and of tables of the exponential and the error functions.

(43) Dr. G. M. Robison: *Divergent double series and sequences.*

For any given double sequence s_{mn}, a new double sequence σ_{mn} may be defined by a linear transformation:

$$\sigma_{mn} = \sum_{k=1,\, l=1}^{m,\, n} a_{mnkl} \cdot s_{kl}.$$

Dr. Robison establishes a necessary and sufficient condition that the transformation be regular, i.e., that when it is applied to a bounded convergent sequence the new sequence shall be convergent to the same value as the original sequence. Further investigations determine when a regular transformation carries a sequence which becomes positively infinite into a sequence which becomes positively infinite. The condition that a transformation transform a bounded convergent sequence (s_{mn}) into a bounded convergent sequence (σ_{mn}), the limits not necessarily being the same, is given; also the condition that a transformation carries a bounded sequence into a bounded convergent sequence. These theorems are also extended to the case where the new sequence is defined as follows:

$$\sigma_{mn} = \sum_{k=1,\, l=1}^{\infty,\, \infty} a_{mnkl} \cdot s_{kl},$$

each element σ_{mn} depending upon all the elements of the original sequence.

(44) Professor G. D. Birkhoff: *An extension of Poincaré's geometric theorem.*

The hypothesis of Poincaré's geometric theorem* requires that the two boundaries of a ring be invariant. As Professor Birkhoff shows, it is sufficient to require the invariance of only one of these boundaries. The extension is very convenient for the dynamical applications.

(45) Dr. J. L. Walsh: *On the location of the roots of polynomials.*

Dr. Walsh proves that if the points $a_1, a_2, \cdots, a_k$ lie on or within a circle whose center is α and radius r_1 and if the points $b_1, b_2, \cdots, b_k$ lie on or within a circle whose center is β and radius r_2, then all the roots of the polynomial

$$f(z) = (z - a_1)(z - a_2)\cdots(z - a_k) \\ - A(z - b_1)(z - b_2)\cdots(z - b_k), \qquad A \neq 1,$$

lie on or within the k circles whose common radius is

$$(r_1 + |A^{1/k}|r_2)/(1 + |A^{1/k}|)$$

and whose centers are the k points $(\alpha - \beta A^{1/k})/(1 - A^{1/k})$, where $A^{1/k}$ takes all the k values possible. If any one of these k circles is exterior to all the others, that circle contains precisely one root of $f(z)$. This theorem holds with a very slight change if $A = 1$. There is a similar theorem for the polynomial $(z - a_1)(z - a_2)\cdots(z - a_k) - A = 0$.

Miss Mullikin was introduced by Professor R. L. Moore, and Dr. Hille by Professor G. D. Birkhoff. In the absence of the authors, the papers of Professor Wilder, Dr. Walsh, Miss Mullikin, Dr. Schweitzer, Dr. Rowe, Mr. Vandiver, Professor Lasley, Professor Moore, and Professor Birkhoff were read by title, and Professor Schwatt's papers were read by Professor H. H. Mitchell. Dr. Zeldin's second paper, Professor Kasner's first, Dr. Wiener's second, and Professor Forsyth's first paper were also read by title.

R. G. D. Richardson,
Secretary.

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* An interesting attempt at a direct demonstration of the theorem has been given recently by Mr. E. Gau, Bulletin des Sciences Mathématiques vol. 43 (1919), p. 12–17. Unfortunately it is tacitly assumed (p. 14) that radial lines are carried over into curves met by any radial line only once.

RECIPROCAL SUBGROUPS OF AN ABELIAN GROUP.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society September 8, 1920.)

§ 1. *Introduction.* Every two subgroups of the group G which have the property that the product of their orders is equal to the order of G have been called *reciprocal subgroups* of G.* A group may have subgroups which have no reciprocals. For instance, the tetrahedral group contains subgroups of order 2 but it does not contain any subgroup of order 6. If a group is abelian, each of its subgroups is known to have at least one reciprocal subgroup. A necessary and sufficient condition that every subgroup of G have one and only one reciprocal is that G be cyclic.

Two invariant subgroups of G will be called *corresponding reciprocal subgroups* of G if each of them is simply isomorphic with the quotient group of G with respect to the other. One of the objects of the present paper is to prove that every subgroup of an abelian group has a corresponding reciprocal subgroup. All the subgroups in a complete set of conjugate subgroups under the group of isomorphisms of G, that is, all the subgroups in a set of I-conjugate subgroups of G, must evidently have the same reciprocal subgroups. Hence the theory of corresponding reciprocal subgroups of an abelian group establishes a correspondence between pairs of sets of I-conjugate subgroups. In what follows it will be assumed that G is abelian.

As G is the direct product of its Sylow subgroups when the order of G is not a power of a prime number p and as the number of I-conjugates of a subgroup of such a G is the product of the numbers of the I-conjugates of the Sylow subgroups of this subgroup it will be assumed in what follows that the order of G is of the form p^m and that G has λ_1 invariants which are separately equal to p^{m_1}, λ_2 invariants which are separately equal to p^{m_2}, $\cdots$, λ_γ invariants which are separately equal to p^{m_γ}. Hence

$$\lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_\gamma m_\gamma = m.$$

It will be convenient to assume that $m_1 > m_2 > \cdots > m_\gamma$.

* This BULLETIN, vol. 9 (1903), p. 541.

Since every subgroup of index p^r contained in G contains the p^rth power of every operator of G, these powers constitute the cross-cut of all the subgroups of index p^r. The quotient group Q_r of G with respect to this cross-cut is simply isomorphic with the characteristic subgroup of G composed of all the operators of G whose orders divide p^r. This characteristic subgroup and the given cross-cut are characteristic corresponding subgroups of G. That is, the subgroup composed of the p^rth power of every operator of G and the subgroup composed of all the operators of G whose orders divide p^r are two characteristic corresponding subgroups of G.

Hence there is a (1, 1) correspondence between the subgroups of G which give rise to quotient groups involving no operators whose orders exceed p^r and the subgroups Q_r. In particular, there is a (1, 1) correspondence between the subgroups of G which give rise to quotient groups of type (1, 1, 1, $\cdots$) and the subgroups of the characteristic subgroup of G generated by all its operators whose orders divide p. Since the latter is known to contain as many subgroups of order p^r as it contains subgroups of index p^r we may state the following theorem.

THEOREM 1. *In every abelian group of order p^m the number of the subgroups of order p^r and of type* (1, 1, 1, $\cdots$) *is equal to the number of the subgroups of index p^r which separately give rise to a quotient group of type* (1, 1, 1, $\cdots$).

In this theorem r has an arbitrary value from 1 to the number of the independent generators of the abelian group in question.

The subgroups of index p^r which separately give rise to a quotient group of type (1, 1, 1, $\cdots$) may be of various types. Hence the theorem stated near the end of the preceding paragraph relates to an enumeration in which no distinction is made between subgroups of somewhat different types. It should be noted that the present method is based on relative properties of subgroups whose largest operators are of a given order and subgroups which give rise to quotient groups involving operators of this order but of no larger order.

If two reciprocal subgroups of G are such that their cross-cut is the identity then the sum of an arbitrary set of independent generators of one of these subgroups and an arbitrary set of independent generators of the other is a set of independent generators of G. Any two such reciprocal sub-

groups are corresponding reciprocal subgroups. A necessary and sufficient condition that two reciprocal subgroups of an abelian group generate this group is that their cross-cut is the identity.

§ 2. *Form of the Number of Subgroups of Certain Types.* It is well known that the number of the subgroups of a given order contained in G is always of the form $1 + kp$. As the subgroups of the same order may be of various types it is of interest to inquire whether there is any general theorem relating to the number of the subgroups of the same type. It is easy to prove that whenever G contains subgroups of the same order but of different types then the number of the subgroups of each of these types except one is divisible by p. That is, every abelian group of order p^m contains one and only one type of subgroups of order p^a, $\alpha < m$, such that the number of the subgroups of this type is of the form $1 + kp$.

To prove this theorem it is only necessary to observe that the number of the subgroups of a given type can be obtained by dividing the number of ways in which a set of independent generators of such a subgroup can be selected from the operators of G by the number of ways in which such a set can be selected from the operators of this subgroup. The numerator and the denominator of this quotient are commonly represented as the product of binomial factors. The first term of such a factor represents the order of the group generated by all the operators of the order in question contained in the group under consideration, while the second term represents the order of the subgroup of the former group composed of its operators which cannot be used as independent generators after the preceding independent generators, if any, have been chosen.

In order that the number of subgroups of a given type be of the form $1 + kp$ it must therefore be necessary that the second term of a factor of the given numerator is always the same as the corresponding second term in the denominator. This implies that the subgroup in question must have the property that if it contains operators of different orders it must involve all the operators of G of the same orders with the possible exception that the operators of highest order found in this subgroup need not include all the operators of the same order found in G. Whenever at least one of these second terms in

the numerator is larger than the corresponding second term in the denominator the number of subgroups is clearly divisible by p. Hence the following theorem has been established.

THEOREM 2. *Whenever an abelian group of order p^m contains subgroups of the same order but of different types then the number of the subgroups of one and of only one of these types is of the form $1 + kp$. The number of the subgroups of each of the other of these types is divisible by p.*

While G contains one and only one type of subgroups of each order which divides p^m such that the number of its subgroups of this type is of the form $1 + kp$ it may contain one or more than one type of subgroups of a given order such that the number of the subgroups of this type is a power of p. It is not difficult to determine a necessary and sufficient condition that the number of the subgroups of a given type be of the form p^a. In fact, one such condition is that each binomial factor of the numerator of the given quotient which represents the number of these subgroups is the product of a power of p by the corresponding factor in the denominator of this quotient. Hence we have the following theorem.

THEOREM 3. *A necessary and sufficient condition that the number of the subgroups of a given type contained in an abelian group G of order p^m be a power of p is that the number of its independent generators of each order increased by the number of its larger independent generators in this set be equal to the number of the independent generators of G whose orders are not less than this order.*

In the special case when there is only one subgroup of a given type the number of these subgroups may be said to be both a power of p and also of the form $1 + kp$. Hence in this special case the conditions involved in the two theorems stated above coincide. In order to illustrate the conditions under which the number of subgroups of a given type is a power of p we shall consider the special case when G is of type $(1, 2, 3, \cdots, m')$. To every combination of one or more of the numbers $1, 2, 3, \cdots, m'$ there corresponds one and only one such type, except the combination which involves all of these numbers. Hence this G contains $2m' - 2$ different types of subgroups besides the identity such that the number of the subgroups of each of these types is a power of p. This is also the number of such types when G has equal invariants but when its distinct invariants are $p, p^2, p^3, \cdots, p^{m'}$.

In general, when the different invariants of G are p^{m_1}, p^{m_2}, $\cdots$, p^{m_λ}, it is not difficult to find the number of the different types of subgroups such that the number of the subgroups of each of these types is a power of p. In every one of the possible combinations of numbers m_1, m_2, $\cdots$, m_λ, each of these numbers may be replaced by every smaller integer which exceeds the one which follows it in this set. The sum of the sets thus obtained is the required number. In particular when G is of type $(m - 1, 1)$ the number of such types of subgroups besides the identity is $2(m - 2)$, and when all the invariants of G are equal to $p^{m'}$ this number is $m' - 1$.

§ 3. *Quotient Groups and their Corresponding Subgroups.* It is well known that every possible quotient group of any abelian group is simply isomorphic with at least one subgroup of this abelian group, but two simply isomorphic subgroups of G do not always give rise to simply isomorphic quotient groups of G. Hence the question arises whether it is possible to associate with an arbitrary subgroup H_1 of G another subgroup H_2 of G such that G/H_1 is simply isomorphic with H_2 and G/H_2 is simply isomorphic with H_1, that is, whether for every subgroup of G there is at least one corresponding reciprocal subgroup.

For the sake of simplicity it will first be assumed that G/H_1 is cyclic, and it will be useful to note the following three possible cases: In the first case at least one operator in the co-set of G which corresponds to a generator of G/H_1 is of the same order as this generator. In the second case the ratios of the order of the smallest operators in a co-set and the order of the corresponding operator in G/H_1 are equal to the same number greater than unity for all the co-sets, excluding the co-set which corresponds to the identity in G/H_1. In the third case, the ratios of these orders are equal to $\rho > 1$ distinct numbers. The significance of conditions which are satisfied in these three cases can easily be determined and may be formulated as follows:

In the first case, any set of independent generators of H_1 together with an arbitrary operator of lowest order in a co-set corresponding to a generator of G/H_1 constitutes a set of independent generators of G. Conversely, whenever a set of independent generators of G can be so chosen that all of them except one generate H_1 then this case will present itself. In this case it is evident that H_1 and the cyclic subgroup gener-

ated by the remaining independent generator of G are corresponding reciprocal subgroups.

The second case implies that a set of independent generators of G can be so selected that all except one of them are independent generators of H_1 while the remaining independent generator of H_1 is a power of the remaining independent generator of G. This generator s is an arbitrary operator of lowest order in a co-set corresponding to a generator of G/H_1. Conversely, whenever a set of independent generators can be so selected that all except one of them are found in H_1 but do not generate H_1, then this second case will present itself. The cyclic subgroup of G which constitutes a corresponding reciprocal subgroup of H_1 is generated by the power of s which is simple isomorphic with G/H_1. It should be noted that the group generated by this power s' does not have only the identity in common with H_1; in fact it may be contained in H_1.

In the third case, the largest number of operators of any set of independent generators of G that can be selected from the operators of H_1 is ρ less than the total number of the independent generators of G, and the ratio of the orders of any two of these ρ generators cannot be less than p^2. On the other hand, whenever the ratio of the orders of any two of ρ generators of G is at least p^2 it is possible to find such a subgroup H_1. In fact, for the independent generators of H_1 we may take the independent generators of G exclusive of these ρ generators, plus the pth power of the largest of these ρ generators multiplied by the next in size, plus the pth power of this second multiplied by the next following in order of magnitude, $\cdots$, plus the pth power of the next to the last of these ρ operators multiplied by the last one. Hence we have the following theorem.

THEOREM 4. *If G is any abelian group of order p^m and ρ is the largest number of operators belonging to a set of independent generators of G and satisfying the condition that the ratio of the orders of any two of these ρ operators is not less than p^2, then G contains a subgroup H_1 which gives rise to a cyclic quotient group and satisfies the condition that at least ρ of the operators of every possible set of independent generators of G are not found in H_1.*

The main element of interest connected with this theorem is the fact that the lowest operators of G which correspond to the various operators of a cyclic quotient group G/H_1 together with the order of this quotient group determine completely

the set of I-conjugate subgroups to which H_1 belongs as well as the set of I-conjugate subgroups of G to which G/H_1 must correspond in order that H_1 and G/H_1 may be corresponding reciprocal subgroups. The ρ or $\rho - 1$ independent generators of H_1 which are not also independent generators of G can be obtained as follows.

First select the ρ independent generators of G $s_1, s_2, \cdots, s_\rho$ which are not contained in H_1 by noting that the first of these generators is any one of the operators of lowest order in the co-set of G which corresponds to a generator of G/H_1. The second is any operator of lowest order in the first lower co-set in which a power of s_1 is not an operator of lowest order. The third is any operator of lowest order in the next lower co-set in which a power of s_2 is not an operator of lowest order. . . . The last is any operator of lowest order in the highest co-set in which a power of $s_{\rho-1}$ is not an operator of lowest order. The independent generators of H_1 in question are then the product of s_2 into the inverse of the power of s_1 which occurs in the same co-set as s_2, the product of s_3 into the inverse of the power of s_2 which occurs in the same co-set as $s_3, \cdots$, the product of s_ρ by x the inverse of the power of $s_{\rho-1}$ which appears in the same co-set is s_ρ, the power of s_ρ which appears in H_1 whenever this power is not equal to the identity.

The cyclic subgroup of G which corresponds to G/H_1 may be obtained as follows: Let $s_1', s_2', \cdots, s_\rho'$ represent the powers of $s_1, s_2, \cdots, s_\rho$ respectively such that these powers are of the same orders as the operators of G/H_1 which correspond to $s_1, s_2, \cdots, s_\rho$ respectively. The product $s_1', s_2', \cdots, s_\rho'$ generates a cyclic subgroup H_1' which is simply isomorphic with G/H_1 and G/H_1' is also simply isomorphic with H_1.

When G/H_1 is non-cyclic it is the direct product of cyclic groups and the preceding arguments apply to these separate cyclic groups. At least one independent generator of G corresponds to each of these cyclic groups. Hence it results that in this case H_1 has again a corresponding reciprocal subgroup, and we have established the following theorem.

THEOREM 5. *Every subgroup of an abelian group has at least one corresponding reciprocal subgroup.*

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PROOF OF AN ARITHMETIC THEOREM DUE TO LIOUVILLE.

BY PROFESSOR E. T. BELL.

1. The theorem, quoted from volume II, page 337, of Dickson's *History of the Theory of Numbers*, is as follows. *If $f(m)$, $F(m)$ are two arbitrary functions having definite values for $m = 1, 2, 3, \cdots$, and*

$$X_\mu(m) = \Sigma d^\mu f(d), \qquad Z_\mu(m) = \Sigma d^\mu F(d),$$

where each summation extends over all divisors d of m, then for any real or complex numbers μ, ν, and $\delta = m/d$, we have

$$(A) \qquad \Sigma d^{\mu-\nu} X_\nu(d) Z_\mu(\delta) = \Sigma d^{\mu-\nu} Z_\nu(d) X_\mu(\delta).$$

No reference to a proof being given, presumably none has been published. But Liouville remarks* that "there is an exceedingly simple method which I shall develop on another occasion, and which will lead us very rapidly by a kind of regular and general algorithm to the formula (A)." Without going into more detail than suffices for the proof of (A) we shall indicate the nature of an algorithm of this type. Arbitrary functions such as f, F which take definite values for integral arguments are called numerical functions All of the functions considered in this paper are of this kind.

2. Let (d, δ) denote any pair of conjugate divisors of m. Form the value of $\varphi_1(x)\psi(y)$, where φ_1 and ψ are arbitrary functions, for $(x, y) = (d, \delta)$, sum $\varphi_1(x)\psi(y)$ over all pairs (d, δ), and denote the result by $\Sigma_m \varphi_1(d)\psi(\delta)$. Let (δ_1, δ_2) denote any pair of conjugate divisors of δ, so that $m = d\delta$; $\delta = \delta_1\delta_2$, $m = d\delta_1\delta_2$, and put $\psi(m) = \Sigma_m \varphi_2(d)\varphi_3(\delta)$; whence

$$\Sigma_m \varphi_1(d)\psi(\delta) = \Sigma_m [\varphi_1(d)\Sigma_\delta \varphi_2(\delta_1)\varphi_3(\delta_2)] = \Sigma_m \varphi_1(d_1)\varphi_2(d_2)\varphi_3(d_3)$$

the last summation extending over all triads (d_1, d_2, d_3) of divisors defined by $m = d_1d_2d_3$. If now (p, q, r), (i, j, k) are any permutations of $(1, 2, 3)$, it is obvious that

$$\Sigma_m [\varphi_p(d)\Sigma_\delta \varphi_q(\delta_1)\varphi_r(\delta_2)] = \Sigma_m \varphi_i(d_1)\varphi_j(d_2)\varphi_k(d_3),$$

which may be written in the purely symbolic form

$$\varphi_p \cdot \varphi_q \varphi_r = \varphi_i \varphi_j \varphi_k.$$

* Liouville, JOURNAL DES MATHÉMATIQUES, (2), vol. 3 (1858), p. 66.

This can be extended to any number of symbolic factors. Thus for four numerical functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, and (p, q, r, s), (i, j, k, l) any permutations of $(1, 2, 3, 4)$, we have

$$\varphi_p\varphi_q\cdot\varphi_r\varphi_s = \varphi_i\cdot\varphi_j\varphi_k\varphi_l = \varphi_i\cdot\varphi_j\varphi_k\cdot\varphi_l = \varphi_i\varphi_j\varphi_k\varphi_l, \text{ etc.}$$

In non-symbolic form, $\varphi_p\varphi_q\cdot\varphi_r\varphi_s$ is

$$\Sigma_m[\Sigma_d\varphi_1(d_1)\varphi_2(d_2)\Sigma_\delta\varphi_3(\delta_1)\varphi_4(\delta_2)],$$

the sums Σ_m, Σ_d, Σ_δ extending to all (d, δ), (d_1, d_2), (δ_1, δ_2), respectively, such that $m = d\delta$, $d = d_1d_2$, $\delta = \delta_1\delta_2$; and $\varphi_i\varphi_j\varphi_k\varphi_l$ is

$$\Sigma_m\varphi_i(d_1)\varphi_j(d_2)\varphi_k(d_3)\varphi_l(d_4),$$

the sum Σ_m extending to all tetrads (d_1, d_2, d_3, d_4) of divisors such that $m = d_1d_2d_3d_4$. The identity of these sums is obvious.

In precisely the same way the φ's in a product $\varphi_1\varphi_2\cdots\varphi_n$ of any number n of symbolic factors are subject to the associative and commutative laws of formal algebra, and the interpretation of these laws for the general case in terms of multiple summations with respect to all the divisors of m distributed into sets of n or fewer each, is evident from the above illustrations.

3. Although it is not required for the proof of (A), we may point out that if each φ is such that $\varphi(1) \neq 0$, there is a unique division in this algebra. For it has been shown elsewhere* that for each φ there is a unique φ' satisfying the equations

$$\Sigma_1\varphi(d)\varphi'(\delta) = 1, \qquad \Sigma_m\varphi(d)\varphi'(\delta) = 0 \qquad (m > 1).$$

Expressing this symbolically we write $\varphi\varphi' = 1$, or $\varphi' = 1/\varphi$, and see that $\varphi_i\varphi_j = \varphi_k$ implies $\varphi_j = \varphi_k\varphi_i'$, or in another form, $\varphi_j = \varphi_k/\varphi_i$. When the arithmetical or algebraic definition of φ is given there is no difficulty in writing down the zeta function† which generates it. The zeta generator of φ' is obviously the reciprocal of the generator of φ.

* Bell, TÔHOKU MATHEMATICAL JOURNAL, vol. 17 (1920), p. 221.

† The zeta function generator of φ is by definition the Dirichlet series in which the coefficient of n^{-s} is $\varphi(n)$; *viz.* it is $\Sigma_{n=1}^{\infty}\varphi(n)/n^s$. For practically all the $\varphi(n)$ existing at present in arithmetic, s can be chosen so that the series converge absolutely, and for an important class of φ's, those in which $\varphi(mn) = \varphi(m)\varphi(n)$ for m, n relatively prime integers >0, the generator is decomposable into factors, etc., cf. Bachmann, *Zahlentheorie*, B. 2 Kap. 11. From these decompositions the relations between the φ's are developed isomorphically to the multiplicative properties of integers by means of the algorithm of this note. Such a course was evidently followed by Liouville in obtaining his results on numerical functions.

4. Write $u_k(m) = m^k$, and put

$$X_k'(m) = X_k(m)/m^k, \qquad Z_k'(m) = Z_k(m)/m^k.$$

Then from the definitions of the functions,

$$X_\mu' = u_{-\mu}f, \qquad Z_\mu' = u_{-\mu}F,$$

and from the associative and commutative laws,

$$u_{-\mu}F \cdot u_{-\nu}f = u_{-\mu}f \cdot u_{-\nu}F,$$

we find $Z_\mu'X_\nu' = X_\mu'Z_\nu'$, which may be written in full as follows:

$$\Sigma_m \frac{Z_\mu(\delta)}{\delta^\mu} \frac{X_\nu(d)}{d^\nu} = \Sigma_m \frac{X_\mu(\delta)}{\delta^\mu} \frac{Z_\nu(d)}{d^\nu}.$$

Multiplying this throughout by m^μ, we get (A).

THE UNIVERSITY OF WASHINGTON,
November 30, 1920.

A SEQUENCE OF POLYNOMIALS CONNECTED WITH THE nTH ROOTS OF UNITY.

BY DR. T. H. GRONWALL.

(Read before the American Mathematical Society September 7, 1920.)

In constructing examples of power series bounded in their circle of convergence and having specified convergence defects on the circle, it is frequently useful to consider polynomials of degree $n - 1$, such that at each of the nth roots of unity, the absolute value of the polynomial is less than or equal to a given constant M. Under these conditions, the maximum absolute value of the polynomial inside or on the unit circle is less than $4M \log n$.*

It is the purpose of this note to determine those polynomials where this maximum is as large as possible. The result may be stated in the following theorem.

THEOREM. *When the polynomial*

$$F(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$$

---

* E. Landau, *Bemerkungen zu einer Arbeit des Herrn Carleman*, MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), pp. 147–153.

has the property that

$$(1)\qquad |F(\epsilon^\nu)| \leqq 1 \qquad (\nu = 0, 1, \cdots, n-1;\ \epsilon = e^{2\pi i/n})$$

and $n > 1$, *then*

$$(2)\qquad |F(z)| < \frac{1}{n}\sum_{\nu=0}^{n-1}\frac{1}{\sin\dfrac{2\nu+1}{2n}\pi}, \qquad \text{for} \qquad |z| \leqq 1,$$

except when $F(z)$ *has the form* $e^{\alpha i}f(\epsilon^{-k}z)$, *where* α *is a real number,* k *is an integer, and*

$$(3)\qquad f(z) = \sum_{\nu=0}^{n-1}\frac{(\epsilon^{-1/2}z)^\nu}{n\sin\dfrac{2\nu+1}{2n}\pi},$$

in which case the upper bound for $|F(z)|$ *is reached when* $z = \epsilon^{k+1/2}$. *The polynomial* $f(z)$ *has all its zeros on the unit circle, one in each of the intervals between two consecutive nth roots of unity, except the interval between* 1 *and* ϵ, *which contains no zero.*

The upper bound given on the right-hand side of (2) *is asymptotically equal to*

$$\frac{2}{\pi}\left(\log n + C + \log\frac{2}{\pi}\right) + o(1),$$

where C *is Euler's constant, and* $o(1)$ *tends to zero as* n *increases indefinitely.*

Let the absolute maximum of the absolute value of $|F(z)|$ for $|z| = 1$ occur between* ϵ^k and ϵ^{k+1}. If we write

$$F(z) = F_1(\epsilon^{-k}z),$$

the absolute maximum of $|F_1(z)|$ for $|z| = 1$ occurs between 1 and ϵ. By (1), we have for $\nu = 0, 1, \cdots, n-1$,

$$F_1(\epsilon^\nu) = M_\nu e^{\alpha_\nu i}, \qquad 0 \leqq M_\nu \leqq 1;$$

and since

$$g(z) = 1 + z + z^2 + \cdots + z^{n-1} = \frac{z^n - 1}{z - 1}$$

* If it occurs at an nth root of unity, then the maximum $|F(z)|$ is less than or equal to unity, which is less than the expression on the right in (2), each sine being less than unity, except the one corresponding to $\nu = (n-1)/2$ when n is odd.

is equal to n for $z = 1$, and is equal to zero for $z = \epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$, we have, by Lagrange's interpolation formula,

$$nF_1(z) = \sum_{\mu=0}^{n-1} M_\mu e^{\alpha_\mu i} g(\epsilon^{n-\mu} z). \tag{4}$$

Now

$$g(\epsilon^{n-\mu} e^{\theta i}) = \frac{e^{n\theta i} - 1}{e^{\left(\theta + \frac{n-\mu}{n}\cdot 2\pi\right)i} - 1}$$

$$= e^{\left(\frac{n-1}{2}\theta - \frac{n-\mu}{n}\pi\right)i} \frac{\sin\frac{n\theta}{2}}{\sin\frac{1}{2}\left(\theta + \frac{n-\mu}{n}\cdot 2\pi\right)},$$

and consequently, if we write $e^{\pi i/n} = \epsilon^{1/2}$,

$$nF_1(e^{\theta i}) = e^{\left(\frac{n-1}{2}\theta + \alpha_0\right)i} \times \tag{5}$$

$$\times \left[M_0 \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} - \sum_{\mu=1}^{n-1} M_\mu e^{(\alpha_\mu - \alpha_0)i} \epsilon^{\mu/2} \frac{\sin\frac{n\theta}{2}}{\sin\frac{1}{2}\left(\theta + \frac{n-\mu}{n}\cdot 2\pi\right)} \right].$$

For $0 < \theta < 2\pi/n$, all the sines in this formula are obviously positive; so that for any value of θ in this interval, we have

$$n|F_1(e^{\theta i})| \leqq \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} + \sum_{\mu=1}^{n-1} \frac{\sin\frac{n\theta}{2}}{\sin\frac{1}{2}\left(\theta + \frac{n-\mu}{n}\cdot 2\pi\right)},$$

where the equality sign holds when and only when we have $M_0 = M_1 = \cdots = M_{n-1} = 1$ and $e^{\alpha_\mu i} = -\epsilon^{-\mu/2} e^{\alpha_0 i}$, that is when $F_1(z) = e^{\alpha_0 i} f_1(z)$, where, by (4) and (5),

$$nf_1(z) = g(z) - \sum_{\mu=1}^{n-1} \epsilon^{-\mu/2} g(\epsilon^{n-\mu} z), \tag{6}$$

$$nf_1(e^{\theta i}) = e^{\frac{n-1}{2}\theta i} \left[\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} + \sum_{\mu=1}^{n-1} \frac{\sin\frac{n\theta}{2}}{\sin\frac{1}{2}\left(\theta + \frac{n-\mu}{n}\cdot 2\pi\right)} \right]. \tag{7}$$

Since $g(z) = 1 + z + \cdots + z^{n-1}$, the coefficient of z^ν in $nf_1(z)$ is, by (6),

$$1 - \sum_{\mu=1}^{n-1} \epsilon^{-\mu/2}\, \epsilon^{\nu(n-\mu)} = 1 - \frac{\epsilon^{-[(1/2)+\nu]} - \epsilon^{-[(1/2)+\nu]n}}{1 - \epsilon^{-[(1/2)+\nu]}}$$

$$= 1 - \frac{\epsilon^{-[(1/2)+\nu]} + 1}{1 - \epsilon^{-[(1/2)+\nu]}}$$

$$= -\frac{2}{\epsilon^{(1/2)+\nu} - 1} = \frac{i\epsilon^{-[(1/4)+(\nu/2)]}}{\sin\dfrac{2\nu+1}{2n}\pi}.$$

Consequently, defining $f(z)$ by (3) and α by $\alpha = \alpha_0 + \pi/2 - \pi/(2n)$, we find $f_1(z) = i\epsilon^{-1/4}f(z)$, and since the absolute maximum of $|f(z)|$ for $|z| = 1$ occurs when all terms to the right in (3) are positive, that is when $z = \epsilon^{1/2}$ or $\theta = \pi/n$ (which is the midpoint of the interval from 0 to $2\pi/n$), it follows that the absolute maximum for $|z| = 1$ of $|F(z)|$ is less than the right-hand member of (2), unless $F_1(z) = e^{\alpha_0 i}f_1(z) = e^{\alpha i}f(z)$, that is when $F(z) = e^{\alpha i}f(\epsilon^{-k}z)$.

The zeros of $f(z)$ are evidently those of $f_1(z)$, and (7) shows that $\varphi(\theta) = e^{-(n-1)\theta i/2}f_1(e^{\theta i})$ is real. Since, by (6), $f_1(1) = 1$, $f_1(\epsilon^\nu) = -\epsilon^{-\nu/2}$ for $\nu = 1, 2, \cdots, n-1$, it follows that $\varphi(0) = 1$, and $\varphi(2\nu\pi/n) = (-1)^{\nu-1}$ for $\nu = 1, 2, \cdots, n-1$, so that $\varphi(\theta)$ has an odd number of zeros in each of the intervals $2\nu\pi/n < \theta < (2\nu + 2)\pi/n$, and consequently $f_1(z)$ has an odd number of zeros on the unit circle in each of the $n - 1$ intervals from ϵ^ν to $\epsilon^{\nu+1}$ ($\nu = 1, 2, \cdots, n-1$). But $f_1(z)$ being of degree $n - 1$ has exactly $n - 1$ zeros, all of which therefore lie on the unit circle, one in each of the intervals mentioned. To find the asymptotic value of the expression to the right in (2), we observe that the νth and $(n - 1 - \nu)$th terms in the sum are equal, and that for n odd, there is a middle term equal to unity. Hence, for n even or odd,

$$\frac{1}{n}\sum_{\nu=0}^{n-1} \frac{1}{\sin\dfrac{2\nu+1}{2n}\pi} = \frac{2}{n}\sum_{\nu=0}^{[n/2]-1} \frac{1}{\sin\dfrac{2\nu+1}{2n}\pi} + o\left(\frac{1}{n}\right),$$

and by the definition of a definite integral

$$\lim_{n=\infty}\frac{\pi}{n}\sum_{\nu=0}^{[n/2]-1}\left(\frac{1}{\sin\dfrac{2\nu+1}{2n}\pi}-\frac{1}{\dfrac{2\nu+1}{2n}\pi}\right)$$
$$=\int_0^{\pi/2}\left(\frac{1}{\sin x}-\frac{1}{x}\right)dx=\log\frac{4}{\pi},$$

or

$$\frac{2}{n}\sum_{\nu=0}^{[n/2]-1}\frac{1}{\sin\dfrac{2\nu+1}{2n}\pi}=\frac{4}{\pi}\sum_{\nu=0}^{[n/2]-1}\frac{1}{2\nu+1}+\frac{2}{\pi}\log\frac{4}{\pi}+o(1).$$

Using the familiar asymptotic formula

$$\sum_{\nu=0}^{m}\frac{1}{2\nu+1}=\frac{1}{2}\log m+\frac{1}{2}C+o(1),$$

where C is Euler's constant, we find

$$\frac{1}{n}\sum_{\nu=0}^{n-1}\frac{1}{\sin\dfrac{2\nu+1}{2n}\pi}=\frac{2}{\pi}\left(\log n+C+\log\frac{2}{\pi}\right)+o(1).$$

TECHNICAL STAFF,
OFFICE OF THE CHIEF OF ORDNANCE

THE MINIMUM AREA BETWEEN A CURVE AND ITS CAUSTIC.

BY PROFESSOR PAUL R. RIDER.

(Read before the American Mathematical Society April 9, 1920.)

If rays from a given source of light are reflected by a curve, the envelope of the rays after reflection is called the caustic of the curve. It is an interesting problem to find the curve which connects two fixed points and which with its caustic and the rays reflected from the fixed points will enclose a minimum area. Euler* proposed and solved a similar problem

* Euler, *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes* or German translation in Ostwald's *Klassiker der exakten Wissenschaften*, no. 46. See also Todhunter, *Researches in the calculus of variations*, chapter 13.

concerning the curve which with its evolute and its normals at the fixed points will enclose the least area. The curves that furnish a solution of Euler's problem are cycloids. The curves that afford a solution of the problem concerning a curve and its caustic are transcendental curves of a more complex type. Their parametric equations are obtained in § 1. The determination of the arbitrary constants that occur in the solution is considered in § 2*. In § 3 a problem is solved which includes the evolute problem and the caustic problem as special cases.

§ 1. *Solution of the Problem.* We shall consider the case in which the source of light is at an infinite distance, and shall assume that the rays are parallel to the y-axis. Let the two fixed points be $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$, $x_0 \neq x_1$. We assume that the curve $y = y(x)$ which joins these two points and which with its caustic and the reflected rays at P_0 and P_1 encloses the minimum area, is single-valued with respect to x, and is continuous and possesses continuous derivatives.

If $\tau = \arctan y'$, the equation of the reflected ray from the point $(x, y(x))$ is readily found to be†

$$Y - y(x) = -\cot 2\tau \cdot (X - x),$$

or

$$(1) \qquad Y - y(x) = \frac{y'^2(x) - 1}{2y'(x)}(X - x),$$

where X, Y are the current coordinates on the reflected ray.

Since the caustic is the envelope of the reflected rays, its parametric equations are obtained in the usual way by differentiating equation (1) with respect to the parameter x, and combining the new equation with (1). The equations of the caustic turn out to be

$$X = x - \frac{y'(x)}{y''(x)}, \quad Y = y(x) + \frac{1 - y'^2(x)}{2y''(x)}.$$

The distance from the point $(x, y(x))$ on the curve to the corresponding point X, Y on the caustic is

$$L = \frac{1 + y'^2(x)}{2y''(x)},$$

* Sufficient conditions, etc., will be treated in a later paper.

† Throughout the paper accents denote differentiation with respect to x.

and the differential of area between the curve and its caustic is

$$dA = \frac{1}{2} L \cos \tau \cdot ds = \frac{1 + y'^2(x)}{4y''(x)} dx,$$

where ds is the element of arc on the curve. Thus our problem reduces to that of minimizing the integral

$$A = \frac{1}{4} \int_{x_0}^{x_1} \frac{1 + y'^2(x)}{y''(x)} dx.$$

By the usual process of the calculus of variations* it is found that if the integral A is to be a minimum, $y(x)$ must satisfy the differential equation

$$\frac{2y'}{y''} + \frac{d}{dx} \frac{1 + y'^2}{y''^2} = \text{const.} = 2a. \tag{2}$$

Moreover, if the direction of the curve $y = y(x)$ at P_0 and P_1 is not prescribed, then we must have $(1 + y'^2)/y'' = 0$ at these points, that is the curve must have cusps at these points, and furthermore the caustic must pass through the points.

If we substitute $\tan \tau$ for y', we can easily reduce equation (2) to the form

$$-\frac{\cos^2 \tau \cdot \tau''}{\tau'^3} = a,$$

from which we find

$$-\frac{\tau''}{\tau'^2} = a \sec^2 \tau \cdot \tau'.$$

Integrating this equation, we get

$$\frac{1}{\tau'} = a \tan \tau + b,$$

or

$$dx = a \tan \tau d\tau + b d\tau.$$

This gives

$$x = a \log \sec \tau + b\tau + c,$$

or, if we set $p = y' = \tan \tau$,

$$x = a \log \sqrt{1 + p^2} + b \arctan p + c. \tag{3}$$

* See Bolza, *Vorlesungen über Variationsrechnung*, p. 152, Ex. 43.

Differentiating (3) with respect to y, we get

$$\frac{1}{p} = \frac{ap+b}{1+p^2}\frac{dp}{dy},$$

from which we find that

$$y = b \log \sqrt{1+p^2} - a \arctan p + ap + d.$$

Thus the parametric equations of the minimizing curves are

$$\begin{cases} x - c = a \log \sqrt{1+p^2} + b \arctan p = a \log \sec \tau + b\tau, \\ y - d = b \log \sqrt{1+p^2} - a \arctan p + ap \\ \qquad\qquad\qquad\qquad = b \log \sec \tau - a\tau + a \tan \tau. \end{cases}$$

The area between a curve of this set and its caustic is

$$\begin{aligned} A &= \frac{1}{4}\int_{x_0}^{x_1} \frac{1+y'^2}{y''}\,dx = \frac{1}{4}\int_{p_0}^{p_1} \frac{(ap+b)^2}{1+p^2}\,dp \\ &= \frac{a^2}{4}(p_1 - p_0) + \frac{ab}{2}\log\sqrt{\frac{1+p_1^2}{1+p_0^2}} \\ &\qquad\qquad + \frac{b^2-a^2}{4}(\arctan p_1 - \arctan p_0) \\ &= \frac{a^2}{4}(\tan\tau_1 - \tan\tau_0) + \frac{ab}{2}\log\frac{\sec\tau_1}{\sec\tau_0} + \frac{b^2-a^2}{4}(\tau_1 - \tau_0). \end{aligned}$$

§ 2. *Determination of the Constants.* If it is prescribed that the curve shall have the slope $p_0 = \tan\tau_0$ at the point $P_0(x_0, y_0)$ and the slope $p_1 = \tan\tau_1$ at the point $P_1(x_1, y_1)$, we have the following four equations from which to determine the constants a, b, c, d:

$$(4) \qquad \begin{cases} x_i = a \log \sqrt{1+p_i^2} + b \arctan p_i + c, \\ y_i = b \log \sqrt{1+p_i^2} - a \arctan p_i + ap_i + d, \quad (i = 0, 1). \end{cases}$$

The determinant of this system is

$$\begin{vmatrix} \log\sqrt{1+p_0^2} & \arctan p_0 & 1 & 0 \\ \log\sqrt{1+p_1^2} & \arctan p_1 & 1 & 0 \\ p_0 - \arctan p_0 & \log\sqrt{1+p_0^2} & 0 & 1 \\ p_1 - \arctan p_1 & \log\sqrt{1+p_1^2} & 0 & 1 \end{vmatrix},$$

which reduces to

$$(p_1 - p_0)(\tau_1 - \tau_0) - (\tau_1 - \tau_0)^2 - (\log \sqrt{1 + p_1^2} - \log \sqrt{1 + p_0^2})^2.$$

Thus if

$$(p_1 - p_0)(\tau_1 - \tau_0) \mp (\tau_1 - \tau_0)^2 + (\log \sqrt{1 + p_1^2} - \log \sqrt{1 + p_0^2})^2,$$

the equations (4) can be solved for a, b, c, d.

§ 3. *A more general Problem.* If the integrand of the integral to be minimized is

$$f = (1 + p^2)^m q^n,$$

where $p = y'$, $q = y''$, we have a type of problem which includes the caustic problem for $m = 1$, $n = -1$, and Euler's evolute problem for $m = 2$, $n = -1$.

It is found* that the minimizing curve must be a solution of the differential equation

$$\frac{2m(1-n)p}{1+p^2} - \frac{n(1-n)q'}{q^2} = \frac{A}{(1+p^2)^m q^n}, \tag{5}$$

in which A is an arbitrary constant. If we multiply both sides of equation (5) by $dp = qdx$, we get

$$\frac{2m(1-n)p}{1+p^2}dp + \frac{n(1-n)}{q}dq = \frac{A}{(1+p^2)^m q^n}dp,$$

which reduces to

$$(1-n)d \log f = \frac{Adp}{f}.$$

Integration gives

$$(1-n)f = Ap + B.$$

Replacing f by $(1 + p^2)^m q^n$ and solving for q (that is dp/dx), we obtain

$$\frac{dp}{dx} = \frac{1}{(1-n)^{1/n}} \frac{(Ap+B)^{1/n}}{(1+p^2)^{m/n}},$$

from which

$$x = (1-n)^{1/n} \int \frac{(1+p^2)^{m/n}}{(Ap+B)^{1/n}} dp. \tag{6}$$

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* See Bolza, loc. cit.

Differentiating with respect to y, we get

$$\frac{1}{p} = (1 - n)^{1/n} \frac{(1 + p^2)^{m/n}}{(Ap + B)^{1/n}} \frac{dp}{dy},$$

and consequently

$$y = (1 - n)^{1/n} \int \frac{(1 + p^2)^{m/n} p}{(Ap + B)^{1/n}} dp. \tag{7}$$

Thus (6) and (7) are the parametric equations of the minimizing curves.

WASHINGTON UNIVERSITY,
ST. LOUIS, MO.,
October 12, 1920.

SHORTER NOTICES.

Il Problema dei Tre Corpi da Newton ai Nostri Giorni. By R. Marcolongo. (Manuali Hoepli.) Milan, Ulrico Hoepli, 1919. 162 pp.

This little book, in the well known style of the Hoepli manuals, presents, as its title indicates, an account of the problem of three bodies from the time of Newton to the present. The author has limited himself strictly to a descriptive account of what has been accomplished in this interval of time, with full references to original memoirs and papers where the interested reader can find the complete developments.

Professor Marcolongo is well known as an authority in the field of dynamical systems, and this book from his pen will be welcomed by all who are interested in the development of mathematical astronomy. Here will be found references to the works of over 200 authors who have contributed to one or more phases of this celebrated problem, together with a short description of the aim, the method of attack, and the results attained.

The book is divided into six chapters. The title of each sufficiently indicates its content.

I. The works of the geometers of the eighteenth century.
II. Reduction of the differential equations to least order.

III. Problem of n bodies. Particular cases.
IV. Uniform algebraic integrals in the problem of n bodies.
V. Approximate solutions by infinite series; by trigonometric series. Researches of Sundman.
VI. The restricted problem of three bodies. Periodic solutions. List of authors cited.

The American student will be interested to find among the authors mentioned the names of Hill, Newcomb, Longley, Brown, Lovett, Macmillan, Moulton, Birkhoff, Wilczynski; and to feel that his own country has not been behind in contributions to this special field of knowledge, important alike in its theoretical aspects and its practical bearings.

The complex development of modern mathematics calls for more books of this type: mathematical Baedeckers, without symbolism, with concise statements of aim, method of attack, and results, and with full references to original sources.

L. W. DOWLING.

Darstellende Geometrie. By Th. Schmid, associate professor of geometry at the technical school of Vienna. Volume I, second edition. (Sammlung Schubert, LXV.) Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger, 1919. 278 pages and 170 figures.

THE first edition of Volume I appeared in 1912, and was reviewed in this BULLETIN (vol. 21 (1914), pp. 204–205.) We are told in the preface to this second edition that the second volume of the first edition has not yet been published. The manuscript is completed, but as the entire edition of the first volume was exhausted, it was decided to publish this second edition of the first volume before proceeding with the second volume.

In the present volume, the exercises appear in smaller type than the text, and are more numerous than in the former edition. At the end of § 12, marked § 35 in the new edition, paragraphs 1 and 2 are interchanged, and four pages of historical and bibliographical matter are added. A similar addition of two pages appears at the end of the volume. Otherwise it is almost a verbatim copy of the first edition.

VIRGIL SNYDER.

NOTES.

The date of the April meeting of the American Mathematical Society at New York has been changed by vote of the Council from April 30 to April 23. At this meeting Professor W. A. Hurwitz will present the opening paper of a symposium on divergent series.

The March number (vol. 22, no. 3) of the ANNALS OF MATHEMATICS contains the following papers: *The asymptotic expansion of the Sturm-Liouville functions*, by F. H. Murray; *On the conformal mapping of a region into a part of itself*, by J. F. Ritt; *Conjugate nets R and their transformations*, by L. P. Eisenhart; *The application of modern theories of integration to the solution of differential equations*, by T. C. Fry.

Volume 55 (1919–1920) of the PROCEEDINGS OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES contains the following mathematical papers: *The functional relation of one variable to each of a number of correlated variables determined by a method of successive approximation to group averages. A contribution to statistical methods*, by G. F. McEwen and E. L. Michael; *Contribution to the general kinetics of material transformations*, by A. J. Lotka; *Rotations in space of even dimensions*, by H. B. Phillips and C. L. E. Moore; *Orbits resulting from assumed laws of motion*, by Arthur Searle; *Some geometric investigations on the general problem of dynamics*, by Joseph Lipka.

The October number (vol. 6, no. 10) of the PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES contains: *On a condition for Helmholtz's equation similar to Lamé's*, by A. G. Webster; *Motion on a surface for any positional field of force*, by Joseph Lipka; The November number contains: *Seminvariants of a general system of linear homogeneous differential equations*, by E. B. Stouffer; *The permanent gravitational field in the Einstein theory*, by L. P. Eisenhart.

The following doctorates in mathematics were recently conferred by the University of London: H. E. J. Curzon: *The reversal of Halphen's transformation;* S. R. U. Saveer: *On the instability of the pear-shaped figure of equilibrium of a rotating mass of homogeneous liquid.*

The following persons have been appointed associate editors of the TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY: Professors O. E. Glenn, A. J. Kempner, H. H. Mitchell, and J. H. M. Wedderburn.

The following persons have been appointed associate editors of the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY for the year 1921: Professors Dunham Jackson, Edward Kasner, D. N. Lehmer, Tullio Levi-Civita, and H. L. Rietz.

At the meeting of the Edinburgh Mathematical Society on December 10, 1920 the following papers were read: by G. D. Stokes, *Analytical treatment of the cam problem;* by D. G. Taylor, *Multiply perspective polygons inscribed in a plane cubic curve;* by T. A. Brown, *Converse of the Le Roy-Lindelöf theorem.* At the meeting on January 14, 1921, the following papers were read: By P. Humbert, *Some extensions of Pincherle's polynomials;* by T. M. MacRobert, *Some asymptotic expressions for the Bessel functions and the Fourier-Bessel expansions.*

At the meeting of the London Mathematical Society on December 9, 1920, the following papers were read: by S. Beatty, *The algebraic theory of algebraic functions of one variable;* by F. Debono, *The construction of magic squares;* by A. S. Eddington, *An application of the calculus of tensors to the theory of finite differences;* by A. R. Forsyth, *Developable surfaces through a couple of guiding curves in different planes;* by J. E. Jones, *The distribution of energy in the neighborhood of a vibrating sphere*; by L. J. Mordell, *I: The reciprocity formula for the Gauss's sums in a quadratic field; II: A new class of definite integrals;* by G. N. Watson, *The product of two hypergeometric functions;* by W. H. Young, *I: Integration over the area of a surface and transformation of the variables in a multiple integral; II: A new set of conditions for a formula for an area.* At the meeting on January 13, 1921, papers were read by A. S. Eddington, *Dr. Sheppard's method of reduction of error by linear compounding;* by W. F. Sheppard, *Conjugate sets of quantities;* by E. A. Milne, *A problem concerning the maxima of certain types of sums and integrals;* by H. J. Priestley, *The linear differential equation of the second order;* by M. Kössler, *The zeros of analytic functions;* by A. C. Dixon, *The theory of a thin elastic plate, bounded by two circular arcs, and clamped;* by G. A. Miller, *Determination of all the characteristic sub-groups of an abelian group.*

The following courses in mathematics are announced:

COLUMBIA UNIVERSITY (summer session, July 5 to August 12).—By Professor Edward Kasner: General survey of modern mathematics, five hours; Mathematical introduction to Einstein's theory of relativity, five hours.—By Professor W. B. Fite: Differential equations, five hours.—By Dr. G. A. Pfeiffer: Theory of functions of a real variable, five hours.—By Dr. J. F. Ritt: Theory of numbers, five hours.

COLUMBIA UNIVERSITY (academic year 1921–1922).—By Professor T. S. Fiske: Differential equations, four hours.—By Professor F. N. Cole: Invariants and higher plane curves, three hours (second term); Theory of groups, three hours.—By Professor D. E. Smith: History of mathematics, two hours; Practicum in the history of mathematics, four hours.—By Professor C. J. Keyser: Modern theories in geometry, four hours; Introduction to mathematical philosophy, two hours (first term).—By Professor Edward Kasner: Einstein's theory of gravitation, two hours.—By Professor W. B. Fite: Infinite series, three hours (first term); Calculus of variations, three hours (second term).—By Dr. G. A. Pfeiffer: The theory of sets of points, three hours.—By Dr. J. F. Ritt: Functional equations, three hours.

CORNELL UNIVERSITY (summer session, July 9 to August 19).—By Professor V. Snyder: Projective geometry, six hours.—By Professor W. A. Hurwitz: Analysis, six hours.

CORNELL UNIVERSITY (academic year 1921–1922).—By Professor J. H. Tanner: Mathematics of finance, two hours (given in each term).—By Professor F. R. Sharpe: Hydrodynamics and elasticity, three hours.—By Professor W. B, Carver: Modern algebra, three hours.—By Professor A. Ranum: Non-euclidean geometry, three hours.—By Professor D. C. Gillespie: Advanced calculus, three hours.—By Professor W. A. Hurwitz: Linear integral and differential equations, three hours.—By Professor C. F. Craig: Functions of a complex variable, three hours.—By Professor F. W. Owens: Theory of probability, three hours.—By Dr. H. M. Morse: Elliptic functions, three hours.—By Dr. Helen B. Owens: Projective geometry, three hours.—By Dr. F. W. Reed: Dynamics, three hours.—By Dr. G. M. Robison: Differential equations, three hours.—By Mr. H. S. Vandiver: Theory of groups, three hours.—By Dr. W. L. G. Williams: Advanced analytic geometry, three hours.

PRINCETON UNIVERSITY (academic year 1921–1922).—By Professor H. B. Fine: Functions of a complex variable, three hours.—By Professor O. Veblen: Projective geometry, three hours.—By Professor J. W. Alexander: Differential equations, three hours. Professors Eisenhart and Veblen will conduct a seminar in relativity.

Professor R. Courant, of the University of Münster, has been called to the chair of mathematics at the University of Göttingen that was vacated by Professor Felix Klein on his retirement.

Professor P. R. Scott Lang, for more than forty years Regius professor of mathematics at the University of St. Andrews, has been knighted.

Professor W. W. Rankin, of the University of North Carolina, who has been on leave of absence for the last two years acting as instructor at Columbia University, has been appointed head of the department of mathematics at Agnes Scott College.

At Cornell University, Professor V. Snyder has been granted leave of absence for the academic year 1921–1922. For the second term of the year he has been awarded a subvention by the Heckscher Research Foundation for the prosecution of research on the theory of algebraic surfaces.

Professor Arthur Searle, Phillips professor of astronomy, emeritus, at Harvard University, died October 24, 1920, at the age of eighty-three years.

Dr. Alexander Pell, formerly professor of mathematics and astronomy and dean of the engineering school at the University of South Dakota, and associate professor at Armour Institute, died January 26, 1921, at the age of sixty-two years. Dr. Pell had been a member of the American Mathematical Society since 1898.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDERSEN (A. F.), BOHR (H.) og MOLLERUP (J.). Cesàro's Summabilitetsmetode med Anvendelse paa Fourier'ske og Dirichlet'ske Rækker. Tre Foredrag holdt i Matematisk Forening (København). København, Gjellerup, 1919. 8vo. 51 + 20 + 25 pp.

BOHR (H.). See ANDERSEN (A. F.).

BOUTROUX (P.). L'idéal scientifique des mathématiciens. Paris, Alcan, 1920. 16mo. 274 pp.

BRENDEL (M.). See KLEIN (F.).

BUHL (A.). Géométrie et analyse des intégrales doubles. (Scientia, No. 36.) Paris, Gauthier-Villars, 1920. 68 pp.

CURZON (H. E. J.). The reversal of Halphen's transformation. London, Constable, 1920. 15 pp. 1s.

FRAENKEL (A.). See KLEIN (F.).

FRASER (D. C.). Newton's interpolation formulas. London, C. and E. Layton (published under the authority of the Institute of Actuaries), 1919. 52 pp. 1s.

HEIJDEMAN (W. J.). Wiskundige hoofstukken dienende als noodzakelijke inleiding tot de differentiaal- en integraalrekening voor den technicus, die de hoogere wiskunde door zelfstudie wil leeren; tevens bevattende de eigenschappen en constructies der kromme lijnen en oppervlakken, welke voor den technicus van belang zijn. Deventer, Kluwer, 1919. 140 pp.

KLEIN (F.), BRENDEL (M.) und SCHLESINGER (L.). Materialien für eine wissenschaftliche Biographie von Gauss. Heft 8: Zahlbegriff und Algebra bei Gauss, von A. Fraenkel. Mit einem Anhang von A. Ostrowski: Zum ersten und vierten Gaussschen Beweise des Fundamentalsatzes der Algebra. Leipzig, Teubner, 1920. 8vo. 60 pp.

LORIA (G.). Newton. Roma, A. F. Formíggini, 1920. 69 pp. L. 3.00

MACH (E.). Die Leitgedanken meiner naturwissenschaftlichen Erkenntnislehre und ihre Aufnahme durch die Zeitgenossen. Sinnliche Elemente und naturwissenschaftliche Begriffe. Zwei Aufsätze. Leipzig, Barth, 1919. 8vo. 31 pp.

MOLLERUP (J.). See ANDERSEN (A. F.).

OSTROWSKI (A.). See KLEIN (F.).

PASCAL (E.). Lezioni di calcolo infinitesimale. Parte 3: Calcolo delle variazioni e delle differenze finite. 2a edizione. (Manuali Hoepli.) Milano, Hoepli, 1918. 11 + 325 pp.

RIABOUCHINSKY (D.). Calcul des valeurs absolues. Copenhagen, 1919. 119 pp.

SCHLESINGER (L.). See KLEIN (F.).

SCHLICK (M.). Allgemeine Erkenntnislehre. Berlin, Springer, 1918. 8vo. 346 pp.

SIBIRANI (F.). See VIVANTI (G.).

VIVANTI (G.). Esercizi di analisi infinitesimale. 2a edizione, accuratamente riveduta ed aumentata con la collaborazione del professore F. Sibirani. Parte 1a. Torino, Lattes, 1920. 7 + 172 pp.

VOLK (O.). Entwicklung der Funktionen einer komplexen Variabeln nach den Funktionen des elliptischen Zylinders. Stuttgart, Grub, 1920. 38 pp.

WATSON (G. N.). See WHITTAKER (E. T.).

WHITTAKER (E. T.) and WATSON (G. N.). A course of modern analysis. 3d edition. Cambridge, University Press, 1920. 7 + 608 pp. 40s.

WITTING (A.). Einführung in die Infinitesimalrechnung. I: Differentialrechnung. 2te Auflage. (Mathematisch-physikalische Bibliothek, Nr. 9.) Leipzig, Teubner, 1920.

II. ELEMENTARY MATHEMATICS.

ARNOLD (E. E.). See DURELL (F.).

BINGHAM (J. A.). See HOWARD (B. H.).

CRELLE (A. L.). Rechentafeln, welche alles Multiplizieren und Dividieren mit Zahlen unter Tausend ganz ersparen, bei grösseren Zahlen aber die Rechnung erleichtern und sicherer machen. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1919.

DAVISON (C.). The elements of plane geometry. Cambridge, University Press, 1920. 280 pp. 10s.

DOOLEY (W. H.). Vocational mathematics. Revised by A. Ritchie-Scott. London, Heath, 1920. 8 + 311 pp. 5s.

DURELL (F.) and ARNOLD (E. E.). A second book in algebra. Chicago, Merrill, 1920. 5 + 330 pp.

HAMILTON (S.). Hamilton's essentials of arithmetic. Lower grades, middle grades, higher grades. Chicago, American Book Company, 1920. 224 + 14, 288 + 23, 320 + 19 pp.

HOWARD (B. H.) and BINGHAM (J. A.). A school geometry. London, Hodder and Stoughton, 1920. 26 + 370 pp. 5s. 6d.

LAYNG (A. E.). Exercises in arithmetic. Arranged in two courses. London, John Murray, 1920. 230 + 31 pp. 3s. 6d.

LEVENTHAL (M. J.) and WEINER (M.). Geometry review book. New York, Review Book Company, 1920. 86 pp.

LINCOLN SCHOOL. Illustrated mathematical talks by pupils of the Lincoln School, New York City. New York, Lincoln School, 1920. 44 pp.

PICKEN (D. K.). See WADDELL (W.).

RITCHIE-SCOTT (A.). See DOOLEY (W. H.).

WADDELL (W.) and PICKEN (D. K.). A first trigonometry. Melbourne, Melville and Mullin, 1920. 7 + 78 pp.

WEINER (M.). See LEVENTHAL (M. J.).

III. APPLIED MATHEMATICS.

ADAMS (O. S.). A study of map projections in general. (U. S. Coast and Geodetic Survey, Special Publication No. 60.) Washington, 1919. 24 pp.

APPELL (P.). Traité de mécanique rationnelle. Tome III: Equilibre et mouvement des milieux continus. 3e édition, entièrement refondue. Tome IV: Figures d'équilibre d'une masse fluide homogène en rotation sous l'attraction newtonienne de ses particles. Paris, Gauthier-Villars, 1920. 8vo. 7 + 646 + 297 pp. Fr. 60.00 + 30.00

ASHFORD (C. E.). Electricity and magnetism; theoretical and practical. 3d edition. London, Arnold, 1920. 12 + 303 pp. 4s. 6d.

BLACKBURN (P. P.). See WHITE (C. J.).

BOUASSE (H.). Pendule, spiral, diapason. Tome I. Paris, Delagrave, 1920. 8vo. 26 + 476 pp.

COTTON (E.). Cours de mécanique générale. Introduction à l'étude de la mécanique industrielle. Tome III: Unités. Travail. Dynamique du point et des systèmes. Grenoble, J. Rey, et Paris, Gauthier-Villars, 1920. 8vo. 138 pp.

DESMARETS (M.). See MALGORN (G.).

DINGLER (H.). Die Grundlagen der Physik. Synthetische Prinzipien der mathematischen Naturphilosophie. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1919. 8vo. 15 + 157 pp.

EILERS (G.). Am Schattenstab. Eine volkstümliche Himmelskunde in geschichtlicher Anordnung. Hamburg, Westermann, 1920. 192 pp.

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THE CHICAGO MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The fifteenth regular Western Meeting of the American Mathematical Society was held at the University of Chicago on Wednesday and Thursday, December 29 and 30, 1920. This meeting was the forty-sixth regular meeting of the Chicago Section, and was held in affiliation with the Convocation week meetings of the American Association for the Advancement of Science.

The total attendance was more than one hundred, including the following eighty-three members of the Society: Professor R. P. Baker, Professor A. A. Bennett, Professor Henry Blumberg, Professor P. P. Boyd, Professor W. D. Cairns, Professor Florian Cajori, Professor J. A. Caparo, Professor R. D. Carmichael, Professor E. W. Chittenden, Professor C. E. Comstock, Professor H. H. Conwell, Professor D. R. Curtiss, Professor W. W. Denton, Professor L. E. Dickson, Professor Arnold Dresden, Professor L. C. Emmons, Professor H. J. Ettlinger, Professor G. C. Evans, Professor T. M. Focke, Professor G. H. Graves, Professor W. L. Hart, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor T. F. Holgate, Professor Dunham Jackson, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor E. P. Lane, Professor A. O. Leuschner, Mrs. M. I. Logsdon, Professor Gertrude I. McCain, Professor W. D. MacMillan, Professor H. W. March, Professor William Marshall, Professor T. E. Mason, Professor L. C. Mathewson, Professor E. D. Meacham, Professor G. A. Miller, Professor W. L. Miser, Professor U. G. Mitchell, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Dr. J. R. Musselman, Professor G. W. Myers, Dr. C. A. Nelson, Mr. H. L. Olson, Professor C. I. Palmer, Professor A. D. Pitcher, Professor L. C. Plant, Professor S. E. Rasor, Professor J. F. Reilly, Professor H. L. Rietz, Professor W. J. Risley, Professor Maria M. Roberts, Professor W. H. Roever, Professor D. A. Rothrock, Dr. A. R. Schweitzer, Professor G. T. Sellew, Professor W. G. Simon, Professor E. B. Skinner, Professor D. E. Smith, Professor E. R. Smith, Professor G. W. Smith, Professor I. W. Smith, Dr. L. L. Steimley, Professor R. B. Stone, Professor E. B.

Stouffer, Professor E. J. Townsend, Dr. B. M. Turner, Professor E. B. Van Vleck, Professor Warren Weaver, Professor W. P. Webber, Mr. F. M. Weida, Professor W. D. A. Westfall, Professor E. J. Wilczynski, Professor C. E. Wilder, Professor D. T. Wilson, Professor R. E. Wilson, Professor C. H. Yeaton, Professor J. W. Young, Professor W. A. Zehring.

The session of Wednesday forenoon was a joint meeting with the Mathematical Association of America and with Sections A and L of the American Association for the Advancement of Science. At this meeting, which was presided over by Professor D. R. Curtiss, chairman of Section A, Professor O. D. Kellogg gave his address entitled *A decade of American mathematics*, as retiring vice-president of Section A of the American Association for the Advancement of Science. His paper was followed by an illustrated paper on *The evolution of algebraic notations*, by Professor Florian Cajori.

The meeting of Wednesday afternoon was held simultaneously with a meeting of the Mathematical Association of America, and was presided over by the chairman of the Chicago Section, Professor R. D. Carmichael. During the Thursday sessions, Professor Carmichael was relieved in the chair by Professor Dunham Jackson, newly-elected vice-president of the Society, and by Professor G. A. Miller.

On Wednesday evening a joint dinner of the Chicago Section, the Mathematical Association of America, the American Astronomical Society, and Sections A and D of the American Association for the Advancement of Science, was held at the Quadrangle Club. At this dinner about one hundred seventy-five persons were present. Professor D. E. Smith, retiring president of the Mathematical Association of America, acted as toastmaster. Short speeches were made by representatives of the different participating organizations. An interesting feature of this dinner consisted of the exhibition by Professor C. I. Palmer of a copy of the 1637 edition of Descartes' Geometry, to which Professor Cajori had alluded in the paper referred to above.

At this meeting the following papers were read:

1. Professor W. P. Webber: *Construction of doubly periodic functions with singular points in the period parallelogram.*

Professor Webber starts with the Weierstrassian p-function and arrives at the function

$$W_2(u) = e^{p(u)} - 1,$$

which is doubly periodic, has zeros at the zeros of $p(u)$ and essential singular points at the poles of $p(u)$. This function is used for constructing more general ones.

It is shown that $W_2(u)$ cannot have an algebraic addition theorem. Among others, the following relations are established: $W_2(u) = [W_2(u) + 1]R(p(u), p'(u))$, where R denotes a rational integral function; $W_2(u) + 1 = e^{F_1(W_2(u), W_2'(u))}$, where F_1 is one of the roots of the cubic in $p(u)$, $4\{p^3 - g_2p - g_3\} = \{W_2'(u)/[W_2(u) + 1]\}^2$, g_2, g_3, being the Weierstrassian invariants for $W_2(u)$. Series are derived for $W_2(u)$.

2. Professor H. J. Ettlinger: *Boundary value problems with regular singular points. Second paper.*

In this paper Professor Ettlinger continues his investigation of the oscillatory properties of the solutions of a second order linear boundary value problem, having regular singular points at the end points of the interval.

3. Professor E. W. Chittenden: *Note on the permutability of functions which have the same Schmidt fundamental functions.*

Let $\varphi_i(x)$, $\psi_i(x)$ denote the normalized Schmidt fundamental functions of a kernel $K(x, y) = \Sigma_{i=1}^{\infty} k_i\varphi_i(x)\psi_i(y)$. Professor Chittenden determines the most general function of the form $H(x, y) = \Sigma_{i=1}^{\infty} h_i\varphi_i(x)\psi_i(y)$ which is permutable of the second kind with $K(x, y)$.

4. Professor E. W. Chittenden: *On kernels which have no Fredholm fundamental functions.*

Professor Chittenden studies the classes of kernels which have no Fredholm fundamental functions. For example, if $\varphi_n(x)$, $\psi_n(x)$, $(n = 0, \pm 1, \pm 2, \cdots)$ is a closed normalized biorthogonal system of functions on an interval $(a \leqq x \leqq b)$, the function $K(x, y) = \Sigma_{n=-\infty}^{+\infty} a_n\varphi_n(x)\psi_{n+p}(y)$ (assuming convergence in mean) where the a_n are all different from zero and p is a positive integer, has no Fredholm fundamental functions, and is not orthogonal on the right or left to any function φ such that $\int_a^b \varphi^2 dx > 0$. In case $a_n = 1/n^2$ and the functions φ_n, ψ_n are continuous, the function $K(x, y)$ is continuous.

5. Professor E. W. Chittenden: *Note on convergence in the mean.*

Professor Chittenden presents an example of a sequence of functions which converges in the mean on an interval ($a \leqq x \leqq b$), but diverges at every point of the interval.

6. Dr. A. R. Schweitzer: *Determination of the spherical transformation in Grassmann's extensive algebra.*

The essence of the geometric application of the quaternions in Hamilton's analysis is to be found in the derivation of Euler's transformation. The attempt of Gibbs to derive these equations by means of Grassmann's Lückenausdrücke appears at a disadvantage when compared with Hamilton's simple deduction based on the transformation qvq^{-1}. Using his article in the MATHEMATISCHE ANNALEN, vol. 69, as a basis, Dr. Schweitzer gives a new derivation of Grassmann's circular transformation and by direct generalization determines the spherical transformation. In this derivation, the simplicity of Hamilton's deduction is completely preserved; it is based on the definitions $X \cdot Q(E_i E_j) = x_1 E_1 Q(E_i E_j) + x_2 E_2 Q(E_i E_j) + x_3 E_3 Q(E_i E_j) + x_4 E_4 Q(E_i E_j)$, $E_k Q(E_i E_j) = Q(E_k E_i) \cdot E_j$, where $i, j, k = 1, 2, 3, 4$, and where the notation is the same as in the article cited above.

7. Dr. A. R. Schweitzer: *On the relation of iterative compositional equations to Lie's theory of transformation groups.*

The theory of Lie's transformation groups may be interpreted as a theory on the solution of certain functional equations of the iterative compositional type, subject to auxiliary conditions. On this basis, the group property arises through the introduction of a suitable notation. In the MATHEMATISCHE ANNALEN, vol. 18, Lie mentions that his one-dimensional, r-parameter functional equations are a generalization of the equation $\phi\{\phi(x, a), b\} = \phi\{x, \phi(a, b)\}$, which Lie seems wrongly to ascribe to Abel, but which is apparently due to C. J. Hill. Dr. Schweitzer shows that a new definition of transformation groups arises by generalizing the functional equation

$$f\{f(x, a), f(b, a)\} = f(x, b)$$

and that certain of his *quasi-distributive* equations are a degenerate case of Lie's n-dimensional, n-parameter functional equations and that certain of his quasi-transitive functional equations are a degenerate case of the quasi-transitive corre-

lative of Lie's n-parameter, n-dimensional functional equations. On the other hand, Dr. Schweitzer's quasi-transitive equations and their inverse correlatives suggest generalizations of Lie's functional equations and his concept transformation group. The memoirs of Schur in the MATHEMATISCHE ANNALEN treating the analytic and non-analytic solutions of Lie's functional equations readily give direction to the research in the analytic and non-analytic solutions of equations in iterative compositions in general.

8. Professor E. J. Wilczynski: *Isothermally conjugate nets.*

The geometric properties which characterize an isothermally conjugate net were, until recently, entirely unknown. One very elegant characterization was given by the late Dr. G. M. Green. But Green overlooked a case in which his criterion does not distinguish between isothermally conjugate nets and other nets of an entirely different character. In this paper, which has appeared in the AMERICAN JOURNAL OF MATHEMATICS, October, 1920, Professor Wilczynski shows how to complete Green's discussion by introducing an important new concept, namely, that of a *pencil* of conjugate nets, at least in the special case of *isothermally* conjugate nets.

9. Professor E. J. Wilczynski: *Transformation of conjugate nets into conjugate nets.*

In this second paper, Professor Wilczynski shows how the projective invariants and covariants of a conjugate net are affected by transformations which change it into a new conjugate net on the same surface. The notion of a pencil of conjugate nets is here developed in its general form, enabling him to simplify in a notable fashion the results of the preceding paper. But there result, at the same time, two other characteristic properties of isothermally conjugate systems which are entirely different in kind from Green's criterion.

10. Professor R. L. Moore: *Conditions under which one of two given closed linear point sets may be thrown into the other one by a continuous transformation of a plane into itself.*

It is easy to show, by the exhibition of examples, that if S_n is a space of one or more dimensions, there exist in S_n two closed, bounded point sets which are in one-to-one continuous correspondence with each other but neither of which can be

thrown into the other by a continuous one-to-one transformation of S_n into itself. Professor Moore raises the question whether, if S_n is an n-dimensional euclidean space lying in an $n + 1$ dimensional euclidean space S_{n+1}, J_1 and J_2 are closed, bounded point sets lying in S_n, and there is a continuous one-to-one correspondence between the points of J_1 and the points of J_2, there exists a continuous one-to-one transformation of S_{n+1} into itself which throws J_1 into J_2. He shows that in the case where $n = 1$ this question can be answered in the affirmative.

11. Professor R. L. Moore: *A closed connected set of points which contains no simple continuous arc.*

Some time ago R. L. Moore and J. R. Kline proposed the following questions: (1) Does there exist a non-degenerate* closed connected set of points which contains no simple continuous arc? (2) Does there exist a non-degenerate connected set of points which contains no arc? In a paper presented to this Society in December, 1919, G. A. Pfeiffer gave an example† showing that the second question may be answered in the affirmative. It is to be noted, however, that not only does the set of points exhibited in this example fail to be closed but it does not even contain a single non-degenerate connected subset that is closed. It, therefore, does not furnish an answer to the first question. In the present paper Professor Moore shows that the first question also may be answered in the affirmative.

12. Professor Florian Cajori: *On the history of symbols for n-factorial.*

Professor Cajori points out the origin of the symbol $\underline{|n}$ for n-factorial and traces the spread of this symbol and of $n!$, particularly in the United States. The paper will be published in Isis.

13. Professor L. E. Dickson: *Homogeneous polynomials with a multiplication theorem.*

In his first paper, Professor Dickson investigates all homogeneous polynomials $f(x_1, \cdots, x_n) \equiv f(x)$ of degree d such that $f(x)f(\xi) \equiv f(X)$, where $X_1, \cdots, X_n$ are bilinear functions

* A set of points is said to be non-degenerate if it contains more than one point.

† Cf. this Bulletin, vol. XXVI (1920), p. 246.

of $x_1, \cdots, x_n$ and $\xi_1, \cdots, \xi_n$. Known examples of f are determinants whose ($n = r^2$) elements are independent variables, norms of algebraic numbers, and sums of 2, 4, or 8 squares. It is convenient to introduce the linear algebra in which the product of $x = \Sigma x_i e_i$ by $\xi = \Sigma \xi_i e_i$ is $X = \Sigma X_i e_i$. It is proved that $f(x)$ must divide the dth powers of the determinants $\Delta(x)$ and $\Delta'(x)$ of the general number of the algebra. It is assumed henceforth that $f(x)$ is not expressible in fewer than n variables. By means of a linear transformation on the x's which leaves $f(x)$ unaltered and one on the ξ's leaving $f(\xi)$ unaltered, we secure the simplification that the new composition $x\xi = X$ takes place in a linear algebra having a principal unit e_1 such that $e_1 x = x e_1 = x$ for every x in the algebra. Now every factor of $f(x)$ admits the composition. Next, if $f(x)$ has a covariant of degree δ and index λ which is not identically zero, then f^δ is divisible by Δ^λ and Δ'^λ. Hence, if $\lambda > 0$, f has the same irreducible factors as Δ and Δ', which also admit the composition. For $n < 5$, the hypothesis is satisfied since the Hessian of f is not identically zero (Gordan and Nöther, MATHEMATISCHE ANNALEN, vol. 10, 1876, p. 564). Further, any covariant of f is then a power of f. But an irreducible quartic surface $f = 0$ whose Hessian is cf^2 is a developable surface whose edge of regression is a twisted cubic curve. By means of these theorems it is shown that, if $n < 5$, f is a product of powers of n linear forms or a power of a quaternary quadratic form. For $n = 5$, the former theorems hold, but the case of an irreducible Δ remains undecided. The paper will appear in the PROCEEDINGS OF THE INTERNATIONAL CONGRESS AT STRASBOURG.

14. Professor L. E. Dickson: *Applications of algebraic and hypercomplex numbers to the complete solution in integers of quadratic diophantine equations in several variables.*

In this second paper, Professor Dickson has obtained formulas giving the complete solution in integers of certain equations, as $x_1^2 + x_2^2 + x_3^2 = x_4^2$, $x_1^2 + \cdots + x_5^2 = x_6^2$, when the parameters take only integral values. Transposing one square, we have to express a product as a sum of 2 or 4 squares. Evident solutions are furnished by the theorem on the norm of a product of two complex integers or two quaternions. When the numbers in the resulting formulas are multiplied by the same number ρ and when ρ is allowed to

take all integral values, the products are shown to give all integral solutions. Use is made of the fact that complex integers obey the laws of arithmetic, and likewise quaternions with integral coordinates provided one of the quaternions used has an odd norm. A like theory applies to $x^2 + y^2 + kz^2 = w^2$, when $k = \pm 2, \pm 3, -5, 7, 11, -13$. But if $k = 5$, the numbers $x + y\sqrt{-5}$ do not obey the laws of arithmetic. Corresponding to the two classes of ideals, there are two distinct sets of formulas which together give all integral solutions. This application of ideals will be developed in a later paper. The remaining topics were presented in detail before the International Congress at Strasbourg and will appear in the PROCEEDINGS OF THE CONGRESS.

15. Professor L. E. Dickson: *Arithmetic of quaternions.*

In this third paper, Professor Dickson recalled that A. Hurwitz (GÖTTINGER NACHRICHTEN, 1896, p. 313) proved that the laws of arithmetic hold for integral quaternions, viz. those whose coordinates are either all integers or all halves of odd integers. Since fractions introduce an inconvenience in applications to Diophantine analysis, it is here proposed to define an integral quaternion to be one whose coordinates are all integers. It is called odd if its norm is odd. It is proved that, if at least one of two integral quaternions a and b is odd, they have a right-hand greatest common divisor d which is uniquely determined up to a unit factor ($\pm 1, \pm i, \pm j, \pm k$), and that integral quaternions A and B can be found such that $d = Aa + Bb$. Similarly there is a left-hand greatest common divisor expressible in the form $a\alpha + b\beta$. The further theory proceeds essentially as in Hurwitz's exposition. The paper has been offered to the PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY.

16. Professor L. E. Dickson: *Determination of all general homogeneous polynomials expressible as determinants with linear elements.*

In the fourth paper by Professor Dickson it is proved that every binary form, every ternary form, every quaternary quadratic form, and a sufficiently general quaternary cubic form can be expressed as a determinant whose elements are linear forms, while no further general form has this property. For a plane curve $f = 0$ of order r, we may assume without

loss of generality that f has no repeated factor. Then some line cuts the curve in r distinct points; take it as the side $z = 0$ of a triangle of reference. As the side $y = 0$, take any line not meeting $z = 0$ at one of its r intersections with the curve. Then for $z = 0$, f is a product of r distinct linear functions $X_i = x + \lambda_i y$. It is proved that f can be expressed in one and but one way as a determinant

$$\begin{vmatrix} X_1 + c_{11}z & z & 0 & \cdots & 0 \\ c_{21}z & X_2 + c_{22}z & z & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{r1}z & c_{r2}z & c_{r3}z & \cdots & X_r + c_{rr}z \end{vmatrix},$$

in which the elements above the diagonal are zero except for the elements z just above it. The paper will appear in the TRANSACTIONS. A related paper, dealing with quaternary cubic forms with attention to rationality, has been offered to the AMERICAN JOURNAL OF MATHEMATICS.

17. Professor G. A. Miller: *I-conjugate operators of an abelian group.*

Two operators of any group G are said to be *I-conjugate* if they correspond in at least one of the possible automorphisms of G. As G is supposed to be abelian, it is possible to find a set of generators of $G(s_1, s_2, \cdots, s_\rho)$ such that the group generated by any arbitrary subset of these ρ operators has only the identity in common with the group generated by the rest of them. Any operator of G expressed in terms of these generators is called *I-reduced* if the number of its constituents which are powers of these operators is as small as possible for the set of I-conjugates to which it belongs. Professor Miller then notes that a necessary and sufficient condition that every I-reduced operator of G involve only one constituent is that the quotient of no two invariants of G exceed p when the order of G is p^m, p being a prime number. He also proves that the smallest groups which involve characteristic operators of odd order are two groups of order 63 and that a necessary and sufficient condition that the quotient groups corresponding to the subgroups generated by two I-reduced operators be of the same type, is that the constituents of these operators generate the same subgroups.

18. Professor W. D. Macmillan: *The integrals*

$$\int_0^x e^{x^2/2}dx, \quad \int_0^x \sin\frac{x^2}{2}\,dx, \quad \int_0^x \cos\frac{x^2}{2}\,dx,$$

and associated divergent series.

It is evident that the integral $\int_0^x e^{x^2/2}dx$ cannot be completely tabulated and thus brought into the category of known functions, because it increases beyond all limits and it has no simple properties which enable us to determine its value for a given large value of x. But if we write $\int_0^x e^{x^2/2}dx = e^{x^2/2}\cdot\varphi(x)$, then $\varphi(x)$, for large values of x, is a decreasing function which has the limit zero. It admits the divergent expansion

$$\varphi(x) \equiv \frac{1}{x} + \frac{1}{x^3} + \frac{1\cdot 3}{x^5} + \frac{1\cdot 3\cdot 5}{x^7} + \cdots,$$

which is very useful for computing the numerical value of φ for large values of x. It is shown by Professor MacMillan that if T_n is the nth term of this series and $\varphi_{n-1}(x)$ is the sum of the first $n - 1$ terms of $\varphi(x)$, and if $2n - 3 < x^2 < 2n - 1$, then T_n is the minimum term and $|\varphi(x) - \varphi_{n-1}(x)| < T_n(x)$; i.e. the error committed in using the divergent series is less than the minimum term provided the series is carried up to, but does not include, the minimum term. Thus, for $x \geqq 6$, the series will give results certainly accurate to 8 decimals, and the order of accuracy increases rapidly with x. A table of values for x up to 6 is given to 7 decimal places. Similar remarks apply to the other integrals.

19. Professor R. P. Baker: *Elementary geometry in n dimensions.*

In this paper, Professor Baker provides a standard method for finding distance and angles between flat spaces in n-dimensional euclidean geometry. Each R_k is supposed given by a matrix $(k + 1)(n + 1)$ of rectangular coordinates with a column of units. The joint invariants of a pair of such flat spaces are (1) a *mutual moment* involving the distance and all the angles, (2) an *inner product* involving all the angles, and (3) a set of *extensionals* of the general type of mutual moments of figures simply constructed from the data. These furnish symmetric functions of the *slopes*. The algebraic equation for determining the slopes always has real roots.

Since the mutual moment of two lines in space is a polarized form of Plücker's identity, the extensionals are polarized forms of identically vanishing relations (extensionals of Plücker's, etc.) of the determinants of the space. The difficulty of dealing with supernumerary coordinates is surmounted by a wide extension of the theorem of Pythagoras. The *content* of any $(k + 1)$ point in R_n is the square root of the sum of the squares of the projections on the axial R_k's. This applies directly to the mutual moment and extensionals and also to the inner product when regression is used to reduce to the case $(k, k, 2k)$. The theory is linear algebra, though regression is used as a device. If progressive methods are desired in connection with the general Pythagorean theorem, we can construct the square of two flat $(k + 1)$ points, which is equivalent to the sum of the square of the projections. Models of an R_3 perspective and development of the *square on the triangle* are shown. Finite dissection, however, fails for tetrahedra, Dehn's condition not being satisfied in general.

20. Professor Dunham Jackson: *Note on an ambiguous case of approximation.*

In recent papers, Professor Jackson has discussed the existence and properties of a trigonometric sum $T_{mn}(x)$ of order n at most, determined by the condition that it shall give the best possible approximation to a given continuous periodic function $f(x)$, in the sense of the integral of the mth power of the absolute value of the error. In these discussions it has been assumed that $m \geqq 1$. The purpose of the present note is to inquire what becomes of the problem for values of m less than 1. It is immediately seen that the problem degenerates for $m \leqq 0$, so that there is occasion to consider only values of m between 0 and 1. It is found that there is always at least one determination of $T_{mn}(x)$ which makes the integral a minimum, but that this determination is not generally unique. Nevertheless, it is possible to treat the question of convergence, when m is held fast and n is allowed to become infinite, in substantially the same way as for $m \geqq 1$. In case there are two or more determinations of $T_{mn}(x)$ for a given value of n, it is immaterial for the convergence which is chosen. The treatment applies equally well to the problem of polynomial approximation, and is in part of still wider application.

21. Professor Dunham Jackson: *On the method of least mth powers for a set of simultaneous equations.*

If there is given a set of p simultaneous linear equations in n unknown quantities, $p > n$, the question may be raised of determining values for the unknowns so that the equations shall be approximately solved, in the sense that the sum of the mth powers of the absolute values of the errors is a minimum. For $m = 2$, this is the classical problem of least squares. Professor Jackson treats the general problem by methods analogous to those used in recent papers on the approximate representation of a function of a continuous variable x, the independent variable now being represented by an index ranging from 1 to p. While there is a general correspondence between the two cases, continuous and discrete, both as to methods and as to results, the parallelism in detail does not seem to be so close as to render a separate treatment of the algebraic problem superfluous.

22. Professor Dunham Jackson: *Note on the convergence of weighted trigonometric series.*

In this note, Professor Jackson shows that a method used recently in connection with certain problems of convergence in the theory of trigonometric and polynomial approximations can be applied to a more general class of problems, including some cases considered on the formal side by Gram (Crelle, vol. 94) in which different weights are assigned to different values of the independent variable.

23. Professor A. J. Kempner: *On polynomials and their residue systems. Second paper.*

In this paper Professor Kempner continues and develops his earlier paper, *On polynomials and their residue systems*, read at the Chicago meeting of the Society in December, 1917. Residue systems of polynomials with respect to a composite numerical modulus are systematically examined.

24. Professor E. R. Smith: *Expansion of the double-frequency function into a series of Hermite's polynomials.*

Dr. Smith gives a development of the general double-frequency function into a series of polynomials which were first investigated by Hermite. The result may be expressed in terms of the successive partial derivatives of the normal

double-frequency function. It is shown that the results are applicable when the regression is non-linear. The generalized form of the equation defining the curve of regression is also discussed.

25. Professor T. E. Mason: *On amicable numbers and their generalizations.*

In this paper Professor Mason presents some new pairs of amicable numbers and some generalized amicable number sets. Dickson defined an amicable k-tuple as k numbers n_1, $n_2, \cdots, n_k$ satisfying the equations

$$S(n_1) = S(n_2) = \cdots = S(n_k) = n_1 + n_2 + \cdots + n_k,$$

where $S(n)$ means the sum of all the divisors of n. Amicable k-tuples are given for $k = 3, 4, 5, 6$. Carmichael defined multiply amicable numbers as numbers m and n satisfying the equations $S(m) = S(n) = t(m + n)$. Multiply amicable number pairs are given for $t = 2, 3$, and multiply amicable triples for $t = 2$.

26. Professor Henry Blumberg: *On the complete characterization of the set of points of approximate continuity.*

It is known that the set C of points where a function is continuous is a ΠI_n, i.e. a product of a denumerable sequence of sets I_n, each of which consists exclusively of inner points; conversely, if a set S is a ΠI_n, then a function exists which is continuous at every point of S and discontinuous elsewhere. If we think of continuity as equivalent to the vanishing of the ordinary saltus (or oscillation, or least upper bound minus greatest lower bound), we are led to the notion of *approximate* continuity of different kinds when, in place of the ordinary saltus, we employ the f-saltus, the d-saltus, the z-saltus, etc., which are obtained by regarding as negligible finite sets, denumerable sets, sets of zero measure, etc., respectively. Hence, we are led to seek a characterization of the sets C_f, C_d, C_z, etc., of points of approximate continuity. Professor Blumberg deals with these and related questions. One of the results is that the sets C_f, C_d, C_z are completely characterized as being expressible in the form ΠI_n.

27. Dr. Gladys E. C. Gibbens: *Comparison of different line-geometric representations for functions of a complex variable.*

In the present paper, Dr. Gibbens generalizes the methods given by Professor Wilczynski (TRANSACTIONS, vol. 20) for

constructing a rectilinear congruence by means of a functional relation between two complex variables. She finds that as long as the planes of the two complex variables remain parallel, the projective properties of the class of congruences defined by means of the totality of all analytic functions $w = F(z)$ are independent of the relative position of the origins, of the angle between the real axes of the two complex variables, and of the distance between the planes. The congruence which corresponds to an individual function $w = F(z)$ in any particular representation corresponds not to itself, but to the function $e^{i\theta}w = F(z)$ if the angle between the real axes of the two planes be changed by θ.

If the planes of the two complex variables are not parallel, the congruence defined by the particular functional relation $w = F(z)$ is projectively equivalent to that defined by the function $e^{i\theta_2}w = F(e^{i\theta_1}z)$, where the planes of the new variables $W = e^{i\theta_2}w$; $Z = e^{i\theta_1}z$ are perpendicular to each other, and where the real axes have been rotated through angles θ_1, θ_2, respectively, in order to become parallel to the line of intersection of the planes. The reality of the focal sheets and the developables of the congruence depends upon the particular function under consideration.

If the two complex variables are projected from a common plane upon concentric spheres, and corresponding points joined, the properties of the resulting congruence are again dependent upon the particular functional relation assumed between the two complex variables. In the last two cases, the generalizations are, so far, inadequate from the point of view of a general theory.

28. Professor Dunham Jackson: *On the trigonometric representation of an ill-defined function.*

The ordinary notion of single-valued function of a variable x can be generalized in the following manner: It may be supposed that the value of y corresponding to any given x is indeterminate, and that any value, within bounds, is to be regarded as possible, but that some values are more probable than others, to an extent indicated by a weight which is a function of x and y, essentially positive or zero, That is, such a function of two variables, $\omega(x, y)$, can be regarded as constituting a sort of function of the single variable x, which is blurred or out of focus, and can be made distinct only by a

more or less arbitrary averaging process. Professor Jackson develops the idea thus vaguely expressed, by a study of certain properties of non-negative functions $\omega(x, y)$. It is shown how the method of least squares, or, more generally, of least mth powers, can be used to determine a definite trigonometric sum, of any prescribed order, which may be regarded as representative of the given indeterminate function, supposed periodic with regard to x; and it is proved that, under fairly general hypotheses, the trigonometric sum thus obtained converges uniformly to a suitably defined single-valued average.

29. Professor E. L. Dodd: *An adaptation of Bing's paradox, involving an arbitrary a priori probability.*

Professor Dodd finds that the essential feature of Bing's paradox remains, even after the justly criticized* constant a priori probability is replaced by an arbitrary non-negative continuous probability $w(x)$. If s men of given age all live a moment, the probability that another man of that age will live a moment is $p = \int_0^1 w(x)x^{s+1}dx / \int_0^1 w(x)x^s dx$.

If the s men all live a year of n moments then the probability that the other man also will live a year is p^n, and this approaches zero with increasing n since $p < 1$. It is practically certain therefore, that the other man will die within the year. It should be noted, however, that the use of the same $w(x)$ for a moment of sixty seconds and for a moment of one second can hardly be justified. Thus n, though it may be large, is not subject to indefinite increase.

30. Professor O. D. Kellogg: *A convergence theorem and an application.*

In this note, Professor Kellogg gives a simple proof of a theorem due to Ascoli on the uniform convergence of a subsequence of equi-continuous functions and makes an application to the proof of the existence of a characteristic function for a symmetric kernel.

31. Professors G. D. Birkhoff and O. D. Kellogg: *Invariant points under transformations in function space.*

If a closed convex region of the plane be mapped into itself by a continuous single-valued transformation, there will be

* Arne Fisher, *The Mathematical Theory of Probabilities*, p. 75, quoting Kroman.

an invariant point. Professors Birkhoff and Kellogg propose to extend this theorem to n dimensions and to the space of continuous functions. Certain existence theorems may be formulated in terms of the invariance of a point in a function space under a transformation, and it is proposed to consider such applications of the theorem.

32. Professor G. C. Evans: *Fundamental points of potential theory.*

Professor Evans treats some of the problems of harmonic functions and monogenic functions by means of Lebesgue-Stieltjes integrals and generalized derivatives.

33. Professor W. L. Hart: *Functionals of summable functions.*

Let H represent the class of all functions $u(x)$ on an interval $a \leqq x \leqq b$ which, together with their squares, are summable in the Lebesgue sense on (a, b). In the first part of his paper, Professor Hart considers real-valued functionals $F[u]$ defined for all u in H. It is assumed that if a sequence u_n of H converges in the mean to a function u, then $\lim_{n\to\infty} F[u_n] = F[u]$. A representation of $F[u]$ in an infinite series is obtained; a mean-value theorem is obtained for the case that $F[u]$ has a differential; an infinite system of functional equations is solved for an unknown function $u(x, s)$, s being a parameter. In the second part of the paper there are considered functionals $G[u; t]$ which for every u of H yield functions of t that belong to H for $c \leqq t \leqq d$. The concepts continuity and differential are defined. The Fourier coefficients of $G[u; t]$ with respect to a complete orthogonal system $[\varphi_1(t), \varphi_2(t), \cdots]$ are found to be functionals of the type $F[u]$. There are obtained for $G[u; t]$ an infinite series converging in the mean, a mean-value theorem, the solution of implicit functional equations, and a treatment of a pseudo-differential functional equation. In many proofs in the paper, fundamental use is made of theorems and definitions relating to functions of infinitely many variables and to pseudo-derivatives, previously treated by the author.

The papers of Professors Dodd, G. C. Evans, and R. L. Moore, and the paper of Dr. Gibbens, were read by title.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

PLEASANT QUESTIONS AND WONDERFUL EFFECTS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 28, 1920.

BY PROFESSOR FRANK MORLEY.

This is for me a day of reckoning. For us mathematicians most of our days are days of reckoning. But usually, to revive an old phrase, we reckon without our host, without consideration of what we owe and to whom we owe it. I feel that I ought at least to thank the host and to contemplate the question of payment. And at once one sees various hosts.

First and foremost the host is this Society, one of the strongholds of idealism in this country. It was a most happy thought when Columbia men formed the scattered pools of mathematical activity into this important organization. No pool suffered a loss of what may be called potential, many gained enormously. The price must have been paid in the increasing work and sacrifice of the early officers. We took what they provided in the cheerful way of youth. And the youthfulness of the mathematician outlives that of most men, at least if he attends a fair number of meetings of this Society. For he can unload his mind if it is overburdened with a problem too hard for him. And he is sure to go away with some idea, grandiose or neat, obtained in that way of chalk and talk which is the easiest way of getting ideas, to a first approximation.

Or, secondly, the host to whom gratitude should be expressed may be the Engelschaar of great men to whom one owes the science as it stands. But this is obviously too big a field. An interesting point here is the effect of unreasoning veneration on elementary teaching, for example the effect of Euclid on elementary geometry or of Euler on elementary algebra. The adjustment of the properly conservative tradition of teaching to admit important applications is no easy matter. But a good guide in the elementary teaching of any subject is the consideration of its immediate usefulness in neighboring regions, its power of trespass.

Thus arithmetic is rightly taught commercially. Algebra

is on the one hand universal arithmetic, on the other hand it is properly limited by its usefulness in geometry. Geometry should be taught with reference to its many uses. Thus every theorem would have an ultimate root in the ground of intelligent human interest. It would have as its motto "Et documenta damus qua simus origine nati."

The attempt to introduce any branch of mathematics as a pure self-contained logical science is bound to disappoint. I do not, of course, mean this to apply to non-elementary teaching. It is proper to separate the sheep from the goats, but I believe it is a mistake to separate the lambs from the kids.

Or, thirdly, the host to whom one is in private duty bound might be the ideas themselves which caught one's mind and helped to form it. These ideas will be, according to taste, logical or musical or practical. For myself, I confess gratitude to certain simple aids like the chess board or the meridians and circles of latitude on a sphere. With the simple background of a chess board in one's mind, one could survive the onslaught of the rabid algebraists and take a sane interest in determinants and in the elliptic functions of those days.

I have gratitude also for the secondary textbooks in that they gave some small point, some footnotes. You saw where a trail left the high road, where adventure began. This helped you on till you met the books which, being outside the curriculum, seemed pure adventure, as for instance, Whitworth's *Choice and Chance* or Casey's *Sequel to Euclid*, or Clifford's *Kinematics*, or the magnificent works of Salmon. Even the careful, learned, and not over-imaginative Todhunter would give hard and stimulating problems. With an attractive problem one hunted through the theory for a means of solution.

Thus the abstract theory of integration might leave one cold, but the finding of area, or centroid, or length of a curve, was a game. This is, I suppose, what lawyers call the case method, but it is just plain normal mental action.

Soon one saw that most decent theories had their applications, and one would take a chance on theory first. Or one would encounter at college a subject like rigid dynamics, where the problems were hurled at one beyond all reason.

It is possible to combine a reverent study of a reasonable number of classics with a decent development of one's own fragment of mind, to combine a respect for cathedrals with a

liking for making mudpies; but for normal human beings the mudpie is the necessary first step to designing a cathedral or to respecting those who do. And our science needs its fair share of the normal-minded, to spread its ideas, to write its popularizing books. It is becoming too narrowly professional. There are not too many memoirs, but too few readable books.

Or, fourthly, the host may be those who have use for mathematics, the physicist or the statistician. Naturally, they would approve of the less formal introductory teaching which I have advocated. A few clear notions of applied mathematics acquired early will bear a fine superstructure, whereas one who tackles the mathematical treatises on (say) electricity, armed merely with a little analysis and less geometry, has a heart-breaking task if he aims at a real mastery of the mathematical side. So that the infusion of concrete applications, whether it is pedagogically sound as I asserted or not, is indispensable if we are to have a school of applied mathematicians. And this I hold to be a chief desideratum.

In this country research in pure mathematics is solidly rooted and actively efflorescent. Next comes the selection and use of the utilizable. This is the work of the applied mathematician. He and the mathematical physicist are on opposite sides of a counter in a drugstore. The latter seeks from the pharmacopeia what he needs; the former, knowing what the store contains, hands him usually not what he asks for, but a guaranteed article just as good. If more connection can be made between the ordered sense of beauty and power of calculation of the mathematicians and the strong intuitive grasp of our physicists and engineers, it is certain that wonderful effects will follow.

Fifth among the hosts should be the one subject which a man, now supposed to be grown, selects as his intellectual home. I venture to speak briefly for geometry on the ground of having worked diligently at relatively simple geometric questions.

That it should be treated logically is all right; but do not vivisect it merely for the sake of logic. By all means let it have foundations; but few will enter thereby. For it is properly the great exemplar of applied mathematics. And thanks to many workers, of whom Einstein is the latest and the most conspicuous, its future will be glorious.

The covariants and contravariants in which the projective

and inversive geometers have delighted, but which they have signally failed to popularize, are now to be almost a commonplace in the wider field of differential geometry. The physicists who would not look at the projective gnats will swallow the differential camels. And there is no doubt more nutriment in a camel. There will be a lively market, and it should be met by some recasting where pedagogic reasons already exist. The mapping of spaces should be led up to by the simplest cases of mapping. Once again, what pedagogy sighed for physical science demands, this time in the field of elementary geometry itself.

If the science should be taught in its early stages not as a jumble of special applications, but always with an honest consideration of its legitimate contexts, then would it still be true of the far wider mathematics of today, that, to quote old Isaac Barrow again, "The Mathematics is the unshaken Foundation of Science and the Plentiful Fountain of Advantage in Human Affairs."

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FALLACIES AND MISCONCEPTIONS IN DIOPHANTINE ANALYSIS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society March 26, 1921.)

§ 1. *Introduction.* Numerous writers have claimed to find all integral solutions of various homogeneous equations when they have actually found merely the rational solutions, expressed by formulas involving rational parameters. They have really left untouched the more difficult problem of finding all the integral solutions exclusively. The fallacies exposed in § 2 and § 3 are merely particular instances of the wide-spread misconception of the problem of solving a homogeneous equation in integers. It is therefore not safe, without reexamination, to place confidence in any claim that a homogeneous equation has been completely solved in integers.

In the next number of this BULLETIN, I shall show how the

theory of ideals can be applied to find all the solutions in integers of the homogeneous equation $x^2 + ay^2 + bz^2 = w^2$.

§ 2. *A Fallacy concerning Pairs of Equations.* It has been regarded as self-evident by all writers,* who have mentioned the topic, that the problem of solving a non-homogeneous equation in rational numbers is equivalent to the problem of solving the corresponding homogeneous equation in integers. Let us examine this question for the particular homogeneous equation

$$(1) \qquad x^2 + 5y^2 = zw$$

and the corresponding non-homogeneous equation

$$(2) \qquad X^2 + 5Y^2 = Z.$$

The problem of solving the latter in rational numbers is trivial. But the problem of solving (1) in integers involves the finding of all divisors of all numbers that can be represented by $x^2 + 5y^2$, which is one of the serious questions in the theory of quadratic forms. This problem will be treated in the next number of the BULLETIN by the theory of ideals; the conclusion is quoted at the end of § 4 below.

It is clear that there must be some fallacy in the customary argument that two such problems are equivalent.† This argument is the following simple one. If x, y, z, w $(w \neq 0)$ are integers satisfying (1) and if we write

$$(3) \qquad \frac{x}{w} = X, \qquad \frac{y}{w} = Y, \qquad \frac{z}{w} = Z,$$

we obtain rational numbers satisfying (2). Conversely, if X, Y, Z are rational numbers satisfying (2), we may express them as fractions (3) with a common denominator and obtain integers x, y, z, w satisfying (1).

Here there is nothing wrong with the algebraic work, nor with the facts deduced. The fallacy lies in the failure to perceive that these facts do not warrant the conclusion that, in the converse case, we have shown how to find all integral solutions. That goal requires that we find all integers w such that the products wX, wY, wZ are integers, viz., x, y, and z. All such integers w are evidently multiples of the minimum

* Including Gauss, *Disquisitiones Arithmeticae*, § 300.

† Namely, that any solution of one equation corresponds to solutions of the other equation under the transformation (3).

positive integer w. To find the minimum w, we need the least common denominator l of the fractions X, Y, and Z. Let d denote the least common denominator of the fractions X and Y, so that $X = \xi/d$, $Y = \eta/d$, where ξ, η, and d are integers without a common factor > 1. Then we have

$$Z = \frac{\xi^2 + 5\eta^2}{d^2}.$$

Before we can find l, we must find the irreducible fraction which equals Z. But this requires the knowledge of all the divisors of all numbers that can be represented by $\xi^2 + 5\eta^2$. Hence we have made no real advance over our initial problem (1) by utilizing our knowledge of the complete solution in rational numbers of the corresponding non-homogeneous equation (2).

§ 3. *The Fallacy when both Equations are Homogeneous.* There is a wide-spread belief that the problem of finding all rational solutions of a homogeneous equation is equivalent to that of finding all its integral solutions. The argument was recently restated by a specialist as follows: (i) the set of all rational solutions contains the set of all integral solutions, and (ii) from the set of all integral solutions it is obvious that the set of all rational solutions is obtained by dividing the numbers in each solution by an arbitrary positive integer.

But remark (i) does not serve the purpose intended, since it leaves unanswered the vital question of how to select the infinitude of integral solutions from the rational solutions. The futility of the argument is emphasized by replacing (i) by the equally trivial remark that all integral solutions occur among the real (or complex) solutions.

In order to bring out clearly the distinction between the two problems, consider the special equation (1). Its rational solutions are obviously all included in the following two types: $x = y = z = 0$, with w any rational number; and x, y, z any rational numbers such that $z \neq 0$, with $w = (x^2 + 5y^2)/z$. We have therefore solved by inspection our first problem of finding all the rational solutions.

Does this information alone serve, as claimed, to yield the complete solution of our second problem of finding all the integral solutions of equation (1)? If so, we should be able, without further theory, to pick out the integral solutions from

the preceding rational solutions. This is easily done for the first type of rational solutions; we have only to restrict w to integral values. For the second type, we must not only restrict x, y, and z to integral values, but we must also examine the condition that $x^2 + 5y^2$ shall be divisible by z. Expressed otherwise, we require a process, valid for arbitrary integers x and y, of finding all divisors z of $x^2 + 5y^2$ (the quotients giving the corresponding values of w). Since we have merely returned to a restatement of our second problem of finding all the integral solutions of (1), we have made no advance whatever on that problem by considering the first problem of finding the rational solutions.

§ 4. *A common Misconception concerning Integral Solutions of a Homogeneous Equation.* To have a concrete case in point, let us express the rational solutions of equation (1) in the customary homogeneous form, which has the advantage of combining into a single formula the two preceding types of solutions. For $z \neq 0$, express x, y, and z as fractions with the positive least common denominator l, and let n be the greatest common divisor of the numerators. Then

$$(4) \qquad x = \frac{na}{l}, \qquad y = \frac{nb}{l}, \qquad z = \frac{nc}{l}, \qquad w = \frac{n(a^2 + 5b^2)}{cl},$$

where a, b, and c are integers without a common factor > 1, while n and l are integers without a common factor > 1, and $cl \neq 0$. Write ρ for $n/(cl)$. Then

$$(5) \qquad x = \rho ac, \qquad y = \rho bc, \qquad z = \rho c^2, \qquad w = \rho(a^2 + 5b^2).$$

The solutions with $z = 0$ have $x = y = 0$ and are of the form (5) with $c = 0$. Hence all rational solutions of (1) are given by (5), in which a, b, and c are integers without a common factor, while ρ is rational.

Some writers are in the habit of suppressing the *proportionality factor* ρ and claiming without further examination that the resulting values give the general solution in integers. Essentially the same error vitiates the claim of Desboves* that he obtains the complete solution in integers of the general homogeneous quadratic equation in n unknowns when one solution $x, y, \cdots$ is given. Since he regarded $mx, my, \cdots$ as the same solution as $x, y, \cdots$, where m is rational, it is clear

* NOUVELLES ANNALES, (3), vol. 3 (1884), pp. 225–39.

that he found at most formulas for the rational* solutions. Thus he deliberately prevented himself from even attacking the far more difficult problem of finding the integral solutions, though he claimed to find them.

For most homogeneous equations the true state of affairs is analogous to what we shall show to be the case for our special equation (1). Unfortunately we do not obtain all integral solutions if we restrict ρ to integral values in (5), but we must employ values whose denominators increase without limit.

By a certain simplification we shall place in its most favorable light the question of describing all sets of numbers a, b, c, and ρ (with a, b, and c integers without a common factor, and ρ rational) for which the solution (5) is integral, and we shall show that there remains an essential difficulty in the determination of these sets. First, if $c = 0$, then $x = y = z = 0$, and w may be identified with any assigned integer k by taking $\rho = k$, $a = 1$, $b = 0$, for example. Next, let $c \neq 0$. Returning from (5) to the equivalent form (4), we see that x, y, z, w are integers if and only if $l = +1$ and $n(a^2 + 5b^2)$ is divisible by c, whence $\rho = n/c$. Eliminating n, we see† that the conditions on a, b, c, and ρ are that ρc and $\rho(a^2 + 5b^2)$ be integers.‡ Hence the infinitude of sets of numbers a, b, c, and ρ for which formulas (5) give integers, and hence give all integral solutions of (1), may be *described* as follows: (i) the sets $a = 1$, $b = c = 0$, with ρ integral; (ii) the sets for which a, b, and c ($c \neq 0$) range over all triples of integers without a common factor, while for each triple ρ ranges over all the irreducible fractions whose denominators are common divisors of c and $a^2 + 5b^2$. But we have not shown how to *determine* the sets a, b, c, and ρ just described. Their determination requires the finding of all divisors of all numbers represented by the quadratic form $a^2 + 5b^2$. Our simplified description of the integral solutions on the basis for the formulas (5) for the

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* But these can be found at once by considering all the lines through the given rational point.

† Also direct from (5) by using the theorem that, if a, b, c have the greatest common divisor 1, integers A, B, C may be found such that $aA + bB + cC = 1$. Multiply by ρc and apply (5). Thus $xA + yB + zC = \rho c =$ integer.

‡ It is now easily proved that the denominators of the ρ's are unlimited. As is known, there is an infinitude of primes p of the form $\alpha^2 + 5\beta^2$. To obtain the solution $x = \alpha$, $y = \beta$, $z = p$, $w = 1$ by (5), we must take the integral factors ρc and c of z to be ± 1 and $\pm p$, whence $\rho = 1/p$, since the choice $\rho c = \pm p$, $c = \pm 1$ would give $\rho = p$, $w > 1$.

rational solutions is therefore no essential improvement upon the description which the proposed equation itself may be said to give.

In accord with the theory to be explained in the next number of this BULLETIN, the successful determination of the integral solutions is made on the basis of a study, not of formulas (5) for the rational solutions, but of the new formulas

$$(6)\qquad \begin{aligned} x &= \rho(ac - 5bd), & z &= \rho(c^2 + 5d^2), \\ y &= \rho(ad + bc), & w &= \rho(a^2 + 5b^2), \end{aligned}$$

which reduce to (5) when $d = 0$ and hence give all the rational solutions. What really happens is well explained in the language of medicine: the injection of the additional integral parameter* d into our solution (5) counteracts the irritation caused by the rational ρ's with their infinitude of denominators. To prevent confusion in a comparison with (6), rewrite (6) in new letters:

$$(7)\qquad \begin{aligned} x &= \sigma(AC - 5BD), & z &= \sigma(C^2 + 5D^2), \\ y &= \sigma(AD + BC), & w &= \sigma(A^2 + 5B^2). \end{aligned}$$

We now attempt to describe all sets of numbers A, B, C, D, and σ (with A, B, C, and D integers without a common factor, and σ rational) for which formulas (7) give integers and hence give all integral solutions of (1). When σ is integral, there is no additional restriction on the integers A, B, C, and D. When σ is an irreducible fraction with the denominator 2, the numbers (7) are all integers if and only if $C \equiv D$, $A \equiv B \pmod 2$. Hence we write

$$D = q,\ C = 2l + q,\ B = r,\quad A = 2n - r,\ \sigma = \tfrac{1}{2}\rho,$$

and we obtain

$$(8)\qquad \begin{aligned} x &= \rho(2ln - lr + nq - 3qr), & y &= \rho(lr + nq), \\ z &= \rho(2l^2 + 2lq + 3q^2), & w &= \rho(2n^2 - 2nr + 3r^2), \end{aligned}$$

* It is rationally redundant. Any given solution (7), which is (6) written in new letters can be expressed in the form (5). If $C = D = 0$, whence $x = y = z = 0$, take $c = 0$ and identify the two w's, which can be done in infinitely many ways. If C and D are not both zero, take

$$a = (AC - 5BD)/t, \quad b = (AD + BC)/t, \quad c = (C^2 + 5D^2)/t, \quad \rho = t\sigma/c,$$

where t is the greatest common divisor of the three numbers whose division by t is indicated. Our former conditions that ρc and $\rho(a^2 + 5b^2)$ be integers now require that σt and $\sigma(A^2 + 5B^2)$ be integers.

where l, q, n, and r are integers without a common factor, and ρ is an integer.* Next, if σ is an irreducible fraction $\rho/5$ with the denominator 5, the numbers (7) are all integers if and only if C and A are divisible by 5. Writing $A = 5b$, $C = -5d$, $B = -a$, $D = c$, we obtain (6). A more typical case is that in which σ is an irreducible fraction of the form $\rho/3$. Then the numbers (7) are all integers if and only if we have $C \equiv D$, $A \equiv \pm B \pmod 3$. Writing $C = \pm D + 3d$, $A = \mp B + 3b$ in (7), we obtain

$$(9) \qquad \begin{aligned} x &= \rho(3bd \pm bD \mp dB - 2BD), \qquad & y &= \rho(bD + dB), \\ z &= \rho(2D^2 \pm 2Dd + 3d^2), & w &= \rho(2B^2 \mp 2Bb + 3b^2). \end{aligned}$$

For the lower signs, we replace D by l, d by $-q$, B by $-n$, b by r, and obtain (8). For the upper signs, we replace D by $l+q$, d by $-q$, B by $r-n$, b by r, and again obtain (8).

In our next paper we shall show how to construct a machine which examines in this manner each of the infinitude of cases corresponding to the values of the denominators of all irreducible fractions σ, and we shall prove that the solutions which result from any denominator are identical with the solutions (6) and (8) which resulted from the denominators 1 and 2. It will then follow that (6) and (8) together give all the integral solutions of (1) when all the parameters take only integral values.

The goal just reached for our example (1) indicates the desirable form for the integral solutions of any homogeneous equation, viz. expressibility by one or more sets of formulas involving only integral parameters. As in our example, the two sets of formulas (6) and (8) which together give all the integral solutions, may be combined, by way of abbreviation, into a single set of formulas (6) in which the denominator of the only non-integral parameter ρ is limited to the values 1 and 2. Conversely, when it is claimed that all integral solutions of a homogeneous equation† are given by formulas

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* Not all solutions (8) are included among solutions (6). When $l = q = n = r = \rho = 1$, (8) gives $x = -1$, $y = 2$, $z = 7$, $w = 3$. If this solution were of the form (6) for integral a, b, c, d, ρ, then $\rho = \pm 1$ by $x = -1$, and, by $z = 7$, $\pm(c^2 + 5d^2) = 7$, which is impossible in integers.

† As to its rational solutions, if we except the rare cases in which recurring series are used, when a homogeneous equation has been completely solved in rational numbers, the unknowns x_1, x_2, $\cdots$ are expressed as homogeneous polynomials f_i, with integral coefficients, of the same degree in certain independent rational parameters A, B, $\cdots$. Writing $A = ka$,

involving a rational parameter, it should be in the sense of an abbreviated statement with explicit indication of easily performable operations leading to formulas containing only integral parameters. Strictly speaking, we do not produce a solution in integers except by a finite number of additions and multiplications performed upon independent integral parameters. These statements are in accord with the evident intention of writers on this subject, even though their conclusions are not proved and very frequently are erroneous.

§ 5. *A Theorem concerning Pairs of Equations.* By way of contrast with § 2, § 3, we note that it is true that the problem of solving any Diophantine equation in rational numbers is equivalent to the problem of solving the corresponding homogeneous equation in rational numbers. In fact, by definition we can pass from the one equation to the other by a substitution like (3). Thus, if x, y, z, w $(w \neq 0)$ give a rational solution of the homogeneous equation, then X, Y, Z give a rational solution of the corresponding equation. Conversely, any rational solution X, Y, Z of the latter gives the solution $x = wX$, $y = wY$, $z = wZ$, w of the homogeneous equation, where w is any rational number. Here there is no delicate question of sorting out solutions of a desired type from those initially obtained.

THE UNIVERSITY OF CHICAGO,
February 15, 1920.

$B = kb$, $\cdots$, where a, b, $\cdots$ are integers without a common factor, we obtain

$$x_1 = \rho f_1(a, b, \cdots), \qquad x_2 = \rho f_2(a, b, \cdots), \ \cdots,$$

where ρ alone takes rational values. If a homogeneous equation in three unknowns represents a unicursal curve (of genus zero), its rational solutions must be of this form, as shown by Hilbert and Hurwitz, ACTA MATHEMATICA, vol. 14 (1890–1), pp. 217–24. Some writers have expressed their solutions as non-homogeneous polynomials in parameters p_1, $\cdots$, p_k; to pass to homogeneous polynomials, we have only to use new parameters P, $P_1 = p_1/P$, $\cdots$, $P_k = p_k/P$. One writer used parameters subject to an equation of condition, which a later writer solved, and passed to independent parameters. The case of dependent parameters is a preliminary stage in the treatment of the problem. For details on these points, with references, see the writer's *History of the Theory of Numbers*, vol. 2 (*Diophantine Analysis*), Carnegie Institution of Washington, 1920, pp. 556–8, 646, 675–6, 695.

ON THE FOURIER COEFFICIENTS OF A CONTINUOUS FUNCTION.

BY DR. T. H. GRONWALL.

(Read before the American Mathematical Society September 8, 1920.)

It is well known that when

$$\frac{a_0}{2}+\sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta)$$

is the Fourier expansion of a function $f(\theta)$ which is real and continuous for $0 \leqq \theta \leqq 2\pi$, then $\Sigma(a_n{}^2 + b_n{}^2)$ converges. Here the exponent 2 cannot in general be replaced by a smaller one; in fact, Carleman* has constructed an example of a continuous $f(\theta)$ where $\Sigma(a_n{}^{2-2\delta} + b_n{}^{2-2\delta})$ diverges for any $\delta > 0$, and this example has been simplified by Landau.†

In the present note it will be shown that, *given any single-valued real function $\varphi(x)$, subject only to the condition that $\varphi(x)$ becomes infinite as x becomes infinite, there exists a real continuous function $f(\theta)$ whose Fourier coefficients a_n, b_n make the series*

$$\sum (a_n{}^2 + b_n{}^2)\varphi\left(\frac{1}{a_n{}^2 + b_n{}^2}\right)$$

divergent. Assuming $\varphi(x) = x^\delta$, where $\delta > 0$, and observing that $(a^2 + b^2)^{1-\delta} < a^{2-2\delta} + b^{2-2\delta}$, we have the particular result referred to above.

If we denote by $f_1(\theta)$ the function conjugate to $f(\theta)$, and write $z = e^{\theta i}$, $F(z) = f(\theta) + if_1(\theta)$, the Fourier expansion of $F(z)$ is $\Sigma_0^\infty c_n z^n$, where $c_0 = a_0/2$, $c_n = a_n - ib_n$ $(n > 0)$. Our statement will be proved by constructing a function $F(z)$ continuous for $|z| = 1$ and such that $\Sigma|c_n|^2\varphi(1/|c_n|^2)$ diverges. This will be done by means of the following result due to Hardy and Littlewood‡ and used by Landau, loc. cit., for a different purpose:

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* T. Carleman, *Ueber die Fourierkoeffizienten einer stetigen Funktion*, ACTA MATH., vol. 41 (1918), pp. 377–384.

† E. Landau, *Bemerkungen zu einer Arbeit des Herrn Carleman*, MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), pp. 147–153.

‡ G. H. Hardy and J. E. Littlewood, *Some problems of diophantine approximation*, ACTA MATH., vol. 37 (1914), pp. 155–239. See p. 220.

Let ξ be a real irrational number such that all the denominators in its expansion in a continued fraction are bounded (for instance $\xi = \sqrt{2}$ or any quadratic irrationality). Then there exists an $A = A(\xi)$ independent of n and z such that for any $n \geqq 1$, and any z on the unit circle $|z| = 1$,

$$\left|\sum_{\nu=1}^{n} e^{\nu^2\pi\xi i} z^\nu\right| < A\sqrt{n}.$$

Making

$$F_\nu(z) = \sum_{\mu=1}^{n_\nu} \frac{e^{\mu^2\pi\xi i}}{\sqrt{n_\nu}} z^\mu,$$

we have therefore $|F_\nu(z)| < A$ for $|z| = 1$; writing $k_\nu = n_0 + n_1 + \cdots + n_{\nu-1}$ and assuming d_ν to be such that $\Sigma|d_\nu|$ converges, we find that the series

$$F(z) = \sum_{\nu=0}^{\infty} d_\nu z^{k_\nu} F_\nu(z)$$

converges uniformly for $|z| = 1$, so that $F(z)$ is continuous on the unit circle. Multiplying by $z^{-n-1}\,dz$ and integrating along the unit circle, we may integrate term by term to the right on account of the uniform convergence, and the Fourier coefficients c_n of $F(z)$ are thus found to be

$$c_n = d_\nu \frac{e^{\mu^2\pi\xi i}}{\sqrt{n_\nu}} \qquad (n = k_\nu + 1,\ k_\nu + 2,\ \cdots,\ k_\nu + n_\nu).$$

Consequently

$$\sum_{n=k_\nu+1}^{k_{\nu+1}} |c_n|^2 \varphi\left(\frac{1}{|c_n|^2}\right) = |d_\nu|^2 \varphi\left(\frac{n_\nu}{|d_\nu|^2}\right),$$

and since $\varphi(x)$ becomes infinite as x becomes infinite, we may choose each n_ν so that

$$\varphi\left(\frac{n_\nu}{|d_\nu|^2}\right) > \frac{|D_\nu|}{|d_\nu|^2},$$

where $\Sigma|D_\nu|$ is any given divergent series. With this choice of n_ν, it follows that $\Sigma|c_n|^2\varphi(1/|c_n|^2)$ diverges, which proves our theorem.

TECHNICAL STAFF,
OFFICE OF THE CHIEF OF ORDNANCE.

EXTENSION OF AN EXISTENCE THEOREM FOR A NON-SELF-ADJOINT LINEAR SYSTEM.

BY PROFESSOR H. J. ETTLINGER.

(Read before the American Mathematical Society December 31, 1919.)

In a recent paper* the writer established the existence of at least one real characteristic number for the non-self-adjoint system

$$(1) \qquad \frac{d}{dx}\left[K(x, \lambda)\frac{du}{dx}\right] - G(x, \lambda)u = 0,$$

$$(2) \quad U_i = A_{i1}u(a) - A_{i2}K(a)u_x(a) \\ - A_{i3}u(b) + A_{i4}K(b)u_x(b) = 0 \quad (i = 1, 2),$$

satisfying the following conditions I, II, III, IV, V, and either VI A or VI B:

I. $K(x, \lambda)$ and $G(x, \lambda)$ are continuous, real functions of x in $a \leqq x \leqq b$ and for all real values of λ in the interval

$$\Lambda(\Lambda_1 < \lambda < \Lambda_2).$$

II. $K(x, \lambda)$ is positive everywhere in (a, b), Λ.

III. The sets of real constants A_{1j} and A_{2j} are not proportional.

IV. For each value of x in (a, b), K and G do not increase as λ increases.

V. $\lim\limits_{\lambda=\Lambda_1} - \dfrac{\min G}{\min K} = -\infty, \quad \lim\limits_{\lambda=\Lambda_2} - \dfrac{\max G}{\max K} = +\infty.$

VI A.† $D_{12}\cdot D_{34} = 0$, together with either

$$(a) \qquad D_{24}^2 + D_{14}^2 \neq 0, \quad D_{14} \geqq 0, \quad D_{24} \geqq 0,$$

or

$$(b) \quad D_{24}^2 + D_{14}^2 = 0, \quad D_{13}^2 + D_{23}^2 \neq 0, \quad D_{13} \leqq 0, \quad D_{23} \leqq 0.$$

VI B. $D_{12}\cdot D_{34} \neq 0$, together with either

$$(a)\ D_{24} > 0 \quad \text{or} \quad (b)\ D_{24} = 0, \quad D_{14} > 0.$$

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* *Existence theorem for the non-self-adjoint linear system of the second order*, ANNALS OF MATHEMATICS, vol. 25 (1920), pp. 278–290.

† $D_{ij} = \begin{vmatrix} A_{1i} & A_{1j} \\ A_{2i} & A_{2j} \end{vmatrix}.$

It is the purpose of this note to extend this theorem to the non-self-adjoint system

$$(1)\qquad \frac{d}{dx}\left[K(x, \lambda)\frac{du}{dx}\right] - G(x, \lambda)u = 0,$$

$$(3)\quad U_i = A_{i1}(\lambda)u(a) - A_{i2}(\lambda)K(a)u_x(a) - A_{i3}(\lambda)u(b) + A_{i4}(\lambda)K(b)u_x(b) = 0 \quad (i = 1, 2),$$

satisfying conditions I, II, IV, V, and the following new conditions: III′, either VI′ A or VI′ B, either VII A or VII B or VII C or VII D or VII E, and VIII.

III′. $A_{1j}(\lambda)$ and $A_{2j}(\lambda)$ are two independent sets of functions continuous in λ throughout Λ.

VI′ A. $D_{12}(\lambda) \equiv 0,\ D_{34}(\lambda) \neq 0$, or $D_{12}(\lambda) \neq 0,\ D_{34}(\lambda) \equiv 0$ together with either

(*a*) $D_{24}^2(\lambda) + D_{14}^2(\lambda) \neq 0,\ D_{14}(\Lambda_1 + \epsilon) \geqq 0,\ D_{24}(\Lambda_1 + \epsilon) \geqq 0,$

or

(*b*) $D_{24}^2(\lambda) + D_{14}^2(\lambda) \equiv 0,\quad D_{13}^2(\lambda) + D_{23}^2(\lambda) \neq 0,$

$$D_{13}(\Lambda_1 + \epsilon) \leqq 0, \qquad D_{23}(\Lambda_1 + \epsilon) \leqq 0.$$

VI′ B. $D_{12}(\lambda) \cdot D_{34}(\lambda) \neq 0$, together with either

(*a*) $$D_{24}(\Lambda_1 + \epsilon) > 0,$$

or

(*b*) $$D_{24}(\lambda) \equiv 0, \quad D_{14}(\Lambda_1 + \epsilon) > 0.$$

VII *A*.

$$D_{24}(\lambda) \neq 0, \quad \Delta\left[\frac{D_{14}(\lambda)}{D_{24}(\lambda)}\right]^* \leqq 0, \quad \Delta\left[\frac{D_{23}(\lambda)}{D_{24}(\lambda)}\right] \leqq 0.$$

VII *B*.

$$D_{14}(\lambda) \neq 0, \qquad \Delta\left[\frac{D_{14}(\lambda)}{D_{24}(\lambda)}\right] \leqq 0,$$

together with either $D_{24}(\lambda) \equiv 0$, or

$$D_{24}(\lambda) \neq 0 \quad \text{and} \quad \Delta\left[\frac{D_{14}(\lambda)}{D_{24}(\lambda)}\right] \leqq 0.$$

VII *C*.

$$D_{23}(\lambda) \neq 0, \qquad \Delta\left[\frac{D_{13}(\lambda)}{D_{23}(\lambda)}\right] \leqq 0,$$

* $\Delta\phi(\lambda) = \phi(\lambda + \Delta\lambda) - \phi(\lambda)$ for $\Delta\lambda > 0$.

together with either $D_{14}(\lambda) \equiv 0$, or

$$D_{14}(\lambda) \neq 0 \quad \text{and} \quad \Delta\left[\frac{D_{13}(\lambda)}{D_{14}(\lambda)}\right] \leqq 0.$$

VII *D*.

$$D_{23}(\lambda) \neq 0 \quad \text{and} \quad \Delta\left[\frac{D_{13}(\lambda)}{D_{23}(\lambda)}\right] \leqq 0,$$

together with either $D_{24}(\lambda) \equiv 0$, or

$$D_{24}(\lambda) \neq 0 \quad \text{and} \quad \Delta\left[\frac{D_{23}(\lambda)}{D_{24}(\lambda)}\right] \leqq 0.$$

VII *E*.* Either $D_{23}(\lambda) \equiv 0$, or

$$D_{23}(\lambda) \neq 0 \quad \text{and} \quad \Delta\left[\frac{D_{13}(\lambda)}{D_{23}(\lambda)}\right] \leqq 0,$$

together with either $D_{24}(\lambda) \equiv 0$, or

$$D_{24}(\lambda) \neq 0 \text{ and } \Delta\left[\frac{D_{14}(\lambda)}{D_{24}(\lambda)}\right] \leqq 0.$$

VIII. $\Delta D_{13} \cdot \Delta D_{24} - \Delta D_{23} \cdot \Delta D_{14} \leqq 0$.

The types of (3) which we shall consider are the following:

Type I. (*a*) $D_{12}(\lambda) \equiv 0, \quad D_{34}(\lambda) \neq 0,$

(*b*) $D_{12}(\lambda) \neq 0, \quad D_{34}(\lambda) \equiv 0.$

Type II. $D_{12}(\lambda) \cdot D_{34}(\lambda) > 0.$

Type III. $D_{12}(\lambda) \cdot D_{34}(\lambda) < 0.$

Then the various sets of non-self-adjoint conditions† are

$$A. \quad \begin{pmatrix} D_{12}(\lambda) & 0 & D_{32}(\lambda) & D_{42}(\lambda) \\ D_{14}(\lambda) & D_{24}(\lambda) & 0 & 0 \end{pmatrix}$$

$$B. \quad \begin{pmatrix} 0 & D_{12}(\lambda) & D_{13}(\lambda) & D_{24}(\lambda) \\ D_{14}(\lambda) & D_{24}(\lambda) & 0 & 0 \end{pmatrix}$$

$$C. \quad \begin{pmatrix} 0 & D_{12}(\lambda) & D_{13}(\lambda) & D_{14}(\lambda) \\ D_{13}(\lambda) & D_{23}(\lambda) & 0 & 0 \end{pmatrix}$$

$$D. \quad \begin{pmatrix} D_{12}(\lambda) & 0 & D_{32}(\lambda) & D_{42}(\lambda) \\ D_{13}(\lambda) & D_{23}(\lambda) & 0 & 0 \end{pmatrix}$$

$$E. \quad \begin{pmatrix} D_{13}(\lambda) & D_{23}(\lambda) & 0 & D_{43}(\lambda) \\ D_{14}(\lambda) & D_{24}(\lambda) & D_{34}(\lambda) & 0 \end{pmatrix}.$$

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* Evidently the alternative conditions VII hold for the corresponding set of conditions enumerated later.

† See ANNALS OF MATHEMATICS, loc. cit.

These conditions ensure the validity of the theorems* used in the proof of the existence theorem stated above, and the proof follows exactly as in the original theorem.

UNIVERSITY OF TEXAS,
July 30, 1920.

ON THE CAUCHY-GOURSAT THEOREM.

BY R. L. BORGER.

(Read before the American Mathematical Society December 30, 1919.)

In order to prove his integral theorem, viz: $\int_C f(z)dz = 0$, Cauchy found it necessary to assume not only that the derivative $f'(z)$ existed but also that it was continuous. Later, proofs were given by Goursat and by Moore† in which the mere existence of $f'(z)$ was shown to be sufficient for the truth of the theorem. These were based upon the analysis of the complex variable.

From the standpoint of the real variable many interesting investigations have developed around the Cauchy-Goursat theorem. They have depended upon Green's theorem. Porter,‡ using the Riemann integral, proved that with proper restrictions upon the component functions, U and V, of the complex function, Green's theorem was true, and hence that Cauchy's integral theorem was also true, even when the derivative $f'(z)$ did not exist. Montel,§ by means of the Lebesgue integral, proved Green's theorem under the hypothesis that U_x, V_y, exist, are bounded, and satisfy the equation

$$U_x = V_y,$$

except at most in a set of measure zero. He was then able to prove the integral theorem, and the existence of the derivative $f'(z)$ for a function of the complex variable $f(z)$ in the closed region considered. The existence, then, of the deriva-

* Cf. *Existence theorems for the general, real, self-adjoint linear system of the second order*, TRANSACTIONS AMER. MATH. SOC., vol. 19 (1918), p. 94.

† TRANSACTIONS AMER. MATH. SOCIETY, vol. 1 (1900).

‡ ANNALS OF MATHEMATICS, (2), vol. 7 (1905–6).

§ ANNALES SCIENTIFIQUES DE L'ECOLE NORMALE SUPÉRIEURE, (3), vol. 24 (1907).

tive is not necessary for the proof of the integral theorem under the restrictions that were imposed upon the partial derivatives by Montel or by Porter.

It is the purpose of this paper to show that Montel's restrictions as to the boundedness of the partial derivatives are not necessary. By means of the Denjoy integral* it will be shown that the line integral of the ordinary Green's theorem vanishes when taken along all closed curves that lie entirely inside the region and that possess a length. From this it follows that it is possible to establish the Cauchy integral theorem without assuming the existence of $f'(z)$.

We shall prove first the following theorem.

THEOREM. *If $U(x, y)$ and $V(x, y)$ are continuous functions of (x, y) which have finite partial derivatives, U_x, V_y, that satisfy the equation*

$$U_x = V_y$$

in each point of a closed region, R, then

$$(Dn)\int_b^{b'} (Dn)\int_a^{a'} U_x(x, y)dxdy,$$

$$(Dn)\int_a^{a'} (Dn)\int_b^{b'} U_x(x, y)dydx$$

exist and are equal for any rectangular region $a \leqq x \leqq a'$, $b \leqq y \leqq b'$ within R.

Since $U_x(x, y)$ is finite in R, we have

$$(1)\qquad (Dn)\int_a^x U_x(x, y)dx = U(x, y) - U(a, y).$$

Then

$$(Dn)\int_b^y (Dn)\int_a^x U_x(x, y)dxdy$$

$$= (Dn)\int_b^y [U(x, y) - U(a, y)]dy$$

exists, since $U(x, y)$ is continuous in (x, y). Similarly, since by hypothesis $U_x = V_y$, we can establish the existence of the integral

* The Denjoy integral will be denoted by the symbol $(Dn)\int$. Concerning its properties, see Denjoy, COMPTES RENDUS, vol. 154 (1912), pp. 859–862; and Hildebrandt, this BULLETIN, vol. 24 (1917), pp. 140–144.

$$(Dn)\int_a^{a'}(Dn)\int_b^{b'}U_x(x, y)dydx.$$

To prove the equality of these two double integrals we prove that

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}\frac{\partial F}{\partial x}, \tag{2}$$

where

$$F(x, y) = (Dn)\int_b^{y}(Dn)\int_a^{x}U_x(x, y)dxdy.$$

Since

$$F(x, y) = (Dn)\int_b^{y}[U(x, y) - U(a, y)]dy,$$

we have

$$\frac{\partial F}{\partial y} = U(x, y) - U(a, y),$$

whence

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial y} = U_x(x, y) = V_y(x, y). \tag{3}$$

Differentiating $F(x, y)$ first with respect to x, we find

$$\frac{\partial F}{\partial x} = \operatorname*{Lim}_{\Delta x=0}\frac{\Delta F}{\Delta x} = \operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^{y}\frac{U(x+\Delta x, y) - U(x, y)}{\Delta x}dy$$

$$= \operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^{y}U_x(x+\theta\Delta x, y)dy.$$

Since $U_x = V_y$, we have

$$\frac{\partial F}{\partial x} = \operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^{y}U_x(x+\theta\Delta x, y)dy$$

$$= \operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^{y}V_y(x+\theta\Delta x, y)dy$$

$$= \operatorname*{Lim}_{\Delta x=0}[V(x+\theta\Delta x, y) - V(x+\theta\Delta x, b)]$$

$$= V(x, y) - V(x, b).$$

It follows that

$$\frac{\partial}{\partial y}\frac{\partial F}{\partial x} = V_y(x, y) = U_x(x, y),$$

whence, by (3),

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\frac{\partial F}{\partial x}. \tag{4}$$

Integrating the left member of (4) with respect to x, we have

$$(Dn)\int_a^{a'} \frac{\partial}{\partial x}\frac{\partial F}{\partial y}\,dx = \frac{\partial F}{\partial y}(a', y) - \frac{\partial F}{\partial y}(a, y)$$
$$= (Dn)\int_a^{a'} U_x(x, y)\,dx,$$

whence

$$(5)\quad (Dn)\int_b^{b'}(Dn)\int_a^{a'} \frac{\partial}{\partial x}\frac{\partial F}{\partial y} = (Dn)\int_b^{b'}\left[\frac{\partial F}{\partial y}(a', y) - \frac{\partial F}{\partial y}(a, y)\right]dy$$
$$= (Dn)\int_b^{b'}(Dn)\int_a^{a'} U_x(x, y)\,dxdy$$
$$= F(a', b') - F(a', b) - F(a, b') + F(a, b).$$

Integrating the right member of (4) with respect to y and then with respect to x, we get

$$(Dn)\int_a^{a'}(Dn)\int_b^{b'} \frac{\partial}{\partial y}\frac{\partial F}{\partial x}\,dydx$$
$$= F(a', b') - F(a', b) - F(a, b') + F(a, b),$$

whence

$$(Dn)\int_b^{b'}(Dn)\int_a^{a'} \frac{\partial}{\partial x}\frac{\partial F}{\partial y}\,dxdy = (Dn)\int_a^{a'}(Dn)\int_b^{b'} \frac{\partial}{\partial y}\frac{\partial F}{\partial x}\,dydx$$

or

$$(6)\quad (Dn)\int_b^{b'}(Dn)\int_a^{a'} U_x(x, y)\,dxdy$$
$$= (Dn)\int_a^{a'}(Dn)\int_b^{b'} U_x(x, y)\,dydx.$$

To prove $\int_C Udy + Vdx = 0$, where C is a closed curve that has a length, we consider first the case in which C is a rectangle. The application of this theorem to the line integral over the rectangle follows from (6) and the equality $U_x = V_y$. We have

$$\int_C Udy + Vdx = 0,$$

where the Denjoy integral reduces to the Riemann integral, since U and V are continuous.

By well known methods* this can be extended to any rectifiable curve bounding a simply connected region. Hence we have the following theorem.

THEOREM. *If U and V are continuous functions of (x, y) possessing finite partial derivatives U_x, V_y, and if*

$$U_x = V_y$$

in a region R, then

$$\int_C U dy + V dx = 0$$

over any rectifiable curve C bounding a simply connected domain lying wholly within R.

The proof of the Cauchy integral theorem is immediate. Moreover if this theorem is satisfied by a continuous function $f(z)$, we know that $f(z)$ is analytic. We may then state our results as follows.

THEOREM. *The necessary and sufficient conditions that $\int f(z)dz = 0$ over any rectifiable curve C bounding a simply connected region lying entirely within a domain R are that the component functions U, V of $f(z) = U + iV$ possess finite partial derivatives in R and that they satisfy the Cauchy-Riemann equations*

$$U_x = V_y, \quad U_y = - V_x.$$

It may be noted that the existence† of $f'(z)$ in any point z_0 requires:

(*a*) The existence of U_x, U_y, V_x, V_y;

(*b*) $U_x = V_y$; $U_y = - V_x$;

$$(c) \quad \Delta U - \left(\frac{\partial U}{\partial x}\Delta x + \frac{\partial U}{\partial y}\Delta\right), \quad \Delta V - \left(\frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial y}\Delta y\right)$$

infinitely small with respect to $|\Delta x| + |\Delta y|$. Hence the hypotheses of the preceding theorem are less restrictive than those of Goursat.‡

OHIO UNIVERSITY,
ATHENS, OHIO.
July 10, 1920.

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* Goursat, *Mathematical Analysis*, vol. II, part 1, p. 67.

† Fréchet, NOUVELLES ANNALES, vol. 19 (1919), p. 219.

‡ TRANSACTIONS, loc. cit.

ON A GENERAL ARITHMETIC FORMULA OF LIOUVILLE.

BY PROFESSOR E. T. BELL.

(Read before the American Mathematical Society April 9, 1921.)

1. *Introduction.* In his *History of the Theory of Numbers* (vol. 2, chap. 11), Dickson gives a resumé of the present state of the celebrated general formulas of Liouville in the theory of numbers, and remarks that the only formula for which no proof has been published is (Q) of the sixth article. Some years ago I developed an analytic method of complete generality suitable for dealing with all such arithmetic questions, and I applied it incidentally to the proofs of Liouville's formulas. By means of this method we may immediately paraphrase any identity between elliptic, abelian or theta functions, provided only that it contains the arguments of the functions and is not merely an identity between constants, into another identity between arithmetic functions of the greatest generality, such, for example, as those considered by Liouville.

A detailed account of the general principles upon which the method is given in my paper in the January number of the TRANSACTIONS. In a continuation of the same paper which is to appear later, there is a selection of illustrative examples including some of Liouville's formulas and others of new kinds. Since the formula (Q) is not among these, however, I shall give a proof of it here. This formula is in fact unique among all of Liouville's, for it is the only one which depends immediately upon the addition theorems for the elliptic functions. It will be necessary first to recall a very special case of a general theorem established in the paper just mentioned.

2. *A general Theorem.* Let $f(x, y)$ denote a function of the arguments x, y which takes a single definite value whenever x and y are positive, zero or negative integers, and let the a, b, c, and d denote integers. Then if the following equation is an identity in u and v, $\Sigma_i a_i \sin b_i u \sin c_i v = 0$, we can infer that $\Sigma_i a_i f(b_i, c_i) = 0$, for $f(x, y)$ defined as above and subjected to the conditions

$$f(x, y) = -f(-x, y) = -f(x, -y), \quad f(0, y) = 0 = f(x, 0).$$

Beyond these conditions $f(x, y)$ is general in the widest sense. A proof of this result will be found in the paper cited above.

3. *The Formula* (*Q*) *of Liouville.* Let us write for brevity $\mu = 2K/\pi$, $k\mu = h$, and consider the following obvious identity, in which accents signify derivatives with respect to u or with respect to v,

$$h\mathrm{sn}\mu(u+v)[(h\mathrm{sn}\mu u)(h\mathrm{sn}\mu v)'-(h\mathrm{sn}\mu v)(h\mathrm{sn}\mu u)']$$
$$=h\mathrm{sn}\mu(u-v)[(h\mathrm{sn}\mu u)(h\mathrm{sn}\mu v)'+(h\mathrm{sn}\mu v)(h\mathrm{sn}\mu u)'].$$

From the classical series for $\mathrm{sn}\mu u$ we have at once

$$h\mathrm{sn}\mu u = 4\Sigma q^{m/2}(\Sigma \sin du), \qquad (h\mathrm{sn}\mu u)' = 4\Sigma q^{m/2}(\Sigma\, d \cos du),$$

where the first summation extends to all $m = 1, 3, 5, \cdots$, and the second to all the divisors $1, d, \cdots, m$ of m. Replacing in the identity the *sn*, *sn'* functions by their equivalent series and then equating coefficients of like powers of q, we find after some easy reductions the following identity in u, v,

$$\Sigma d_3[\sin (d_1+d_3)u \sin (d_1+d_2)v-\sin (d_1+d_2)u \sin (d_1+d_3)v$$
$$+\sin (s_1-d_3)u \sin (d_1+d_2)v-\sin (d_1+d_2)u \sin (d_1-d_3)v]$$
$$=\Sigma d_3[\sin (d_1+d_3)u \sin (d_1-d_2)v-\sin (d_1-d_2)u \sin (d_1+d_3)v$$
$$+ \sin (d_1-d_3)u \sin (d_1-d_2)v-\sin (d_1-d_2)u \sin (d_1-d_3)v],$$

where the summations refer to all positive divisors d_1, d_2, d_3 defined by

$$m = m_1 + m_2 + m_3, \quad m_1 = d_1\delta_1, \quad m_2 = d_2\delta_2, \quad m_3 = d_3\delta_3,$$

m, m_1, m_2, m_3 being odd and positive and m constant. Hence by the theorem quoted in § 2,

$$\Sigma d_3[f(d_1 + d_3, d_1 + d_2) - f(d_1 + d_2, d_1 + d_3)$$
$$+ f(d_1 - d_3, d_1 + d_2) - f(d_1 + d_2, d_1 - d_3)]$$
$$= \Sigma d_3[f(d_1 + d_3, d_1 - d_2) - f(d_1 - d_2, d_1 + d_3)$$
$$+ f(d_1 - d_3, d_1 - d_2) - f(d_1 - d_2, d_1 - d_3)].$$

Now putting $\psi(x, y) = f(x, y) - f(y, x)$, we see that $\psi(x, y)$ has a single determinate value whenever x and y are positive, zero or negative integers, and that $\psi(x, y)$ satisfies the conditions

$$\psi(- x, y) = - \psi(x, y) = \psi(x, - y) = \psi(y, x),$$
$$\psi(0, y) = 0 = \psi(x, 0).$$

Moreover, if $\psi(x, y)$ is the most general function satisfying all of these conditions, we can put $\psi(x, y) = f(x, y) - f(y, x)$ without loss of generality. A brief discussion of such questions is given in section III of the paper cited. Hence from the identity for $f(x, y)$, we infer at once that

$$\Sigma d_3[\psi(d_1 + d_3, d_1 + d_2) + \psi(d_1 - d_3, d_1 + d_2)]$$
$$= \Sigma d_3[\psi(d_1 + d_3, d_1 - d_2) + \psi(d_1 - d_3, d_1 - d_2)],$$

and this is the formula (Q) of Liouville.

The 39 forms of the addition theorems given by Jacobi in section 18 of the *Fundamenta Nova* imply a multitude of such consequences, many of which are of arithmetic interest.

The author wishes to express his indebtedness to Professor Frank Nelson Cole for encouragement and inspiration, not only in this paper, but for much of his other mathematical work.

UNIVERSITY OF WASHINGTON,
January 19, 1921.

SHORTER NOTICES

General Theory of Polyconic Projections. By Oscar S. Adams. Washington, United States Coast and Geodetic Survey, 1919. Special Publication No. 57. 174 pp.

There are many ways of representing, or projecting, the surface of the earth, or parts of it, upon a plane. Any system of lines may be chosen to represent the parallels of latitude, and a second system to represent the meridians. The book before us is designed to give a full account of the so-called polyconic projection, that is the projection in which parallels of latitude are represented by arcs of a non-concentric system of circles with collinear centers. The line of centers is usually, but not necessarily, taken for the central, or principal, meridian. The mathematical problem consists in setting up the equations for the meridians under various hypotheses, methods for constructing the meridians, spacing the parallels, determining the magnification, and so forth. These details the author has worked out for various cases, deriving the formulas for the ellipsoid as well as for the sphere.

Stereographic projection is one type of polyconic projection.

The author devotes considerable space to it, taking for the plane of projection the equatorial plane, or the plane of a meridian, or the plane of the horizon of a given place.

Another important case is that in which a given parallel is represented by a circle whose radius is equal to an element of the tangent cone to the earth's surface along the parallel and included between the point of contact and the vertex. Parallels are spaced along the central meridian in proportion to their true distances along this meridian. This is the case usually referred to as polyconic projection. (See the article on *Mathematical Geography* by Col. Sir A. R. Clarke in the ENCYCLOPAEDIA BRITTANICA.) This projection has been used extensively by the U. S. Coast and Geodetic Survey.

The book might be improved considerably by numbering the formulas and by making the important ones stand out more prominently than is done in the text. For example, the equation for a meridian in the general polyconic projection is derived on page 12 without comment as to its basic importance, indeed without stating that it is the equation for a meridian.

Those who are interested in the practical construction of maps will no doubt find the book of great assistance, and to these it must make its appeal.

L. W. DOWLING.

Einleitung in die Mengenlehre. By A. Fraenkel. Berlin, J. Springer, 1919. iv + 155 pages.

The author proposes to give an introduction to the theory of infinite sets which can be understood by anyone who has sufficient interest and patience. No other prerequisites are set down. The author tells that he had experience in this sort of presentation during the recent war, when he lightened many wearisome hours by explaining *Mengenlehre* to his comrades in the field.

Among the chapter headings are: the concept of set; the concepts of equivalence and infinite sets; countable sets; the continuum; the concept of cardinal number; comparability of cardinal numbers; operations on cardinal numbers; ordered sets and types of order; linear point sets; well-ordered sets, well-ordering and its significance; logical paradoxes and the concept of set.

The choice of topics and the extent to which each topic is treated are well determined. Scientific honesty is not sacrificed

for (apparently) easy assimilation by the reader. Of course, the treatment of many of the above topics must be incomplete in a presentation such as is given in this book, but the author points out any incompleteness in each definition or proof, and suggests possible ways of filling the lacunae.

Some of the exposition is prolix, but prolixity is difficult to avoid in an exposition designed for the general reader. Even with this prolixity the book is very readable. Many examples well illustrate the abstract treatment of the various topics. Some of the more detailed and technical material which may be omitted without destroying the continuity of the exposition is printed in small type.

In the last chapter the author becomes dogmatic in some statements concerning the principle of selection (das Auswahlprinzip) of Zermelo; but enough is said to enable one to see that the author's point of view is not the only possible one. The axiomatic setting up of the theory of sets according to Zermelo, the paradoxes which are to be avoided in this way, and the bearing of the problem of well-ordering on these matters, are explained here quite clearly, considering the limitations imposed by a popular exposition of these abstruse subjects. The method of *logicizing*, and more particularly the theory of types of Russell, are not mentioned, although a footnote reference to Russell's books is given.

The book should be very useful for upper collegiate classes in mathematics and for those interested in mathematical philosophy in a general way. It should help to introduce to a wider circle the ideas and methods of a fundamental and interesting branch of mathematics.

G. A. PFEIFFER.

Descriptive Geometry. By Ervin Kenison and Harry Cyrus Bradley. New York, The Macmillan Company, 1917. vii + 287 pp.

This is one of a series of texts on topics in engineering edited by E. R. Hedrick. In their preface the authors state, "This book represents a teaching experience of more than twenty years on the part of both the authors at the Massachusetts Institute of Technology. . . . The point of view · · · is · · · that of the draftsman. Mathematical formulae and analytic computations have been almost entirely suppressed. . . . The method of attack throughout the book

is intended to be that which shall most clearly present the actual conditions in space. . . . The amount of ground covered is that which is considered sufficient to enable the student to begin the study of the technical drawings of any line of engineering or architecture. It is not intended to be a complete treatise on descriptive geometry. Detailed exposition of such branches as shades and shadows, perspective, stereographic projection, axonometry, the solution of spherical triangles, etc., will not be found."

Questions relating to points, straight lines, and planes occupy the first three-fifths of this text; while the latter part, including twenty-two of the fifty-two problems, is devoted to tangent lines and planes, and to curved surfaces and their intersections. The first two problems,—to find the traces of a straight line, and to find its projections from its traces,—are on pages 24 and 25. The third problem,—to find the true length of a straight line,—is found thirty pages further along, after four chapters on simple shadows, the representation of the plane, the profile plane of projection, and secondary planes of projection. After a chapter on simple intersections and developments, there follow twelve problems grouped into three chapters on lines in a plane and parallel lines and planes, on perpendicular lines and planes, and on the intersections of planes and of lines and planes. The fifteen problems involving the revolution and counter-revolution of planes follow a chapter on the intersection of planes and solids.

Of the ten problems on tangent planes there are three each for cones and cylinders, and two each for spheres and double-curved surfaces of revolution. Then follow problems on the intersection of a plane with a cone, a frustum of a cone, a cylinder, a prism or a pyramid, and a double-curved surface of revolution. The book is concluded by problems on the intersection of two cylinders, of a cylinder and a cone, of two cones, of a sphere and a cone, of two surfaces of revolution whose axes intersect, and of any two curved surfaces.

These problems are carefully worked out with figures, analysis, construction, general case, and special cases. They will probably bring to the student an idea of the general methods employed. The one criticism that the reviewer would suggest is the absence of exercises to be worked out by the student. The book gives, however, an interesting presentation and is quite worth careful study.

E. B. Cowley.

NOTES.

The January number (vol. 43, no. 1) of the AMERICAN JOURNAL OF MATHEMATICS contains the following papers: *Multiple binary forms with the closure property*, by A. B. Coble; *Einstein's theory of gravitation: determination of the field by light signals*, by Edward Kasner; *Note on Einstein's equation of an orbit*, by Frank Morley; *A one-to-one representation of geodesics on a surface of negative curvature*, by H. M. Morse; *Conjugate systems with indeterminate axis curves*, by E. P. Lane.

NATURE has published a special number (vol. 106, no. 2677, February 17, 1921) devoted entirely to discussions of various aspects of relativity theory. It contains papers by Professor A. Einstein, Mr. E. Cunningham, Sir F. Dyson, Dr. A. C. D. Crommelin, Dr. C. E. St. John, Professor G. B. Mathews, Mr. J. H. Jeans, Professor H. A. Lorentz, Sir O. Lodge, Professor H. Weyl, Professor A. S. Eddington, Dr. N. Campbell, Miss Dorothy Wrench and Mr. H. Jeffreys, and Professor H. W. Carr.

The National Research Council requested Professor H. L. Rietz, of the University of Iowa, to call together a small group for a preliminary conference to discuss the desirability of a committee of the Council in the field of the applications of mathematics to statistics, and to make appropriate recommendations. This group consisted of Professors J. W. Glover, E. V. Huntington, Raymond Pearl, and W. M. Persons, and the conference was held in Washington, January 21–22.

The gold medal of the Royal Astronomical Society has been awarded to Professor H. N. Russell, for his contributions to the theory of stellar evolution.

The Paris Academy of Sciences has awarded 3000 francs from its Loutreuil Foundation to Henri Brocard and Léon Lemoyne, for the publication of the second and third volumes of their work entitled *Courbes géométriques remarquables planes et gauches*.

At the University of Toulouse, Professor Buhl has at his own request been transferred from the chair of general mathematics to that of the differential and integral calculus.

Canon J. M. Wilson has been elected president of the British Mathematical Association.

At the Massachusetts Institute of Technology, Assistant Professor Joseph Lipka has been granted leave of absence for the year 1921–1922. He intends to study in Rome.

At Cornell University Mr. H. S. Vandiver has been awarded a subvention from the Heckscher Research Foundation to support investigations on the theory of algebraic numbers.

At Westminster College, New Wilmington, Pa., Associate Professor J. V. McKelvey, of Iowa State College, has been appointed Professor.

Professor Georges Humbert, of the Ecole Polytechnique and the Collège de France, member of the Paris Academy of Sciences in the section of geometry, died January 22, 1921, at the age of sixty-two years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

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Bjorling (C. F. E.). Oevningsuppgifter i differentialkalkyl och algebraisk analys. Lund, Gleerup, 1920. 8vo. 63 pp.

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Bolzano (B.). Paradoxien des Unendlichen. Mit Unterstützung der Gesellschaft zur Förderung Deutscher Wissenschaft, Kunst und Literatur in Böhmen herausgegeben von der Philosophischen Gesellschaft an der Universität Wien durch Alois Höfler. Mit Anmerkungen versehen von Hans Hahn. (Philosophische Bibliothek, Band 99.) Leipzig, Meiner, 1921. 12mo. 12 + 157 pp.

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BURALI-FORTI (C.). Fondamenti per la geometria del triangolo. Palermo, Capozzi, 1919.

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PÀL (J.). See BOHR (H.).

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SCHREK (D. J. E.). Beginselen der analytische meetkunde. Groningen, Noordhoff, 1919. 8vo. 144 pp.

TURC (A.). Introduction élémentaire à la géométrie Lobatschewskienne. Ouvrage posthume, publié d'après les notes de l'auteur. Genève, Kündig, 1914. 8vo. 170 pp.

VIVANTI (G.). Lezioni di analisi infinitesimale. 2a edizione. Parte 1a. Torino, Lattes, 1920. 8vo. 7 + 693 pp. L. 50.00

DE VRIES (H). "De vierde dimensies," eene inleiding tot de vergelijkende studie der verschillende meetkunden. Groningen, Noordhofi, 1915. 8vo. 12 + 142 pp.

WEYL (H.). See RIEMANN (B.).

WILLERS (A.). Graphische Integration. (Sammlung Göschen, Nr. 801.) Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1920.

II. ELEMENTARY MATHEMATICS.

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FRIIS-PETERSEN (F.) og Jessen (J. L. W.). Aritmetik og Algebra for det toaarige Præliminærkursus. 8vo. København, Gjellerup, 1919–1920. 63 + 63 pp.

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HARVEY (F. W.). Everyman's mathematics. London, Methuen, 1920. 138 pp. 4s.

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III. APPLIED MATHEMATICS.

ADAMS (O. S.). See BOWIE (W.).

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BOQUET (F.). Tables du mouvement képlérien. 1ère partie. Paris, Hermann, 1920. 8vo. 6 + 205 pp.

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BOUASSE (H.). Pendule, spiral, diapason. Tome 2. Paris, Delagrave, 1920. 8vo. 22 + 518 pp.

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DÉVÉDEC (P.). Application de la résistance des matériaux au calcul des ouvrages de béton armé. Paris, Dunod, 1920. 8vo. 16 + 372 pp. Fr. 40.00

EINSTEIN (A.). Ueber die spezielle und die allgemeine Relativitätstheorie. 6te Auflage. Braunschweig, Vieweg, 1920. 8vo. 4 + 83 pp.

FORREST (S. M.). Mathematics for technical students. Junior course. With answers. London, Arnold, 1920. 8 + 260 pp.

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GUILLOT (L.). Cours de mécanique. Tome 1. 2e édition, revue et mise à jour. Paris et Liège, Béranger, 1920. 8vo. 451 pp. Fr. 36.00

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STUYVAERT (M.). Statique. Dynamique. Ghent, Van Rysselberghe et Rombaut, 1920. 8vo. 205 pp. Fr. 20.00

WEYL (H.). Raum, Zeit, Materie. Vorlesungen über allgemeine Relativitätstheorie. 3te, umgearbeitete Auflage. Berlin, Springer, 1920.

YOUNG (A. E.). Some investigations in the theory of map projections. London, Royal Geographical Society, 1920. 8 + 76 pp.

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The two hundred fourteenth regular meeting of the Society was held at Columbia University on Saturday, February 26, extending through the usual morning and afternoon sessions. The attendance included the following thirty-five members:

Professor J. W. Alexander, Mr. D. R. Belcher, Professor A. A. Bennett, Professor E. W. Brown, Dr. G. A. Campbell, Dr. Tobias Dantzig, Dr. Jesse Douglas, Professor L. P. Eisenhart, Professor H. B. Fine, Mr. R. M. Foster, Mr. Philip Franklin, Dr. T. C. Fry, Professor O. E. Glenn, Dr. T. H. Gronwall, Dr. C. C. Grove, Dr. A. A. Himwich, Professor E. V. Huntington, Mr. S. A. Joffe, Professor Edward Kasner, Professor O. D. Kellogg, Dr. E. A. T. Kircher, Dr. K. W. Lamson, Mr. Harry Langman, Professor G. W. Mullins, Professor F. W. Owens, Dr. E. L. Post, Professor H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Miss Louise E. C. Stuerm, Professor H. D. Thompson, Professor E. B. Wilson.

Ex-President H. B. Fine occupied the chair. The Council announced the election of the following one hundred fourteen persons to membership in the Society:

Professor Orrin Wilson Albert, Grinnell College;
Professor David Robert Allen, University of Utah;
Mr. Elbert Frank Allen, University of Missouri;
Professor Charles Lincoln Arnold, Ohio State University;
Miss Mary Caroline Ball, Northwestern University;
Professor Wightman Samuel Beckwith, Ohio Northern University;
Mr. William Noël Birchby, California Institute of Technology;
Dr. Edwin Mortimer Blake, Brooklyn, N. Y.;
Miss Rachel Blodgett, Harvard University;
Professor Rosser Daniel Bohannan, Ohio State University;
Professor John David Bond, Agricultural and Mechanical College of Texas;
Mr. Frederick William Borgward, Syracuse University;
Dr. Henry Roy Brahana, University of Illinois;
Professor William Mayo Brodie, Virginia Polytechnic Institute;
Mr. Bancroft Huntington Brown, Harvard University;
Professor Lillian Olive Brown, Hood College;
Professor Frank Newton Bryant, Syracuse University;
Miss Minnie Wilford Caldwell, Hardin College;
Miss Evelyn Teresa Carroll, Wells College;

Mr. Ming-Cheng Chow, University of Cincinnati;
Mr. Oliver Charles Collins, University of Nebraska;
Professor Allan Ray Congdon, University of Nebraska;
Miss Mary Lucile Copenhover, University of Oregon;
Miss Julia Dale, University of Missouri;
Mr. Harold Thayer Davis, University of Wisconsin;
Miss Alice Crowell Dean, Rice Institute;
Professor Dewey Stevens Dearman, Millsaps Academy;
Professor Alexander Dillingham, United States Naval Academy;
Professor Eleanor Catherine Doak, Mount Holyoke College;
Mr. Theodore Doll, Northwestern University;
Rev. Joseph Nicholas Donohue, Notre Dame University;
Mr. Finis Omer Duncan, University of Missouri;
Miss Edna May Feltges, University of Wisconsin;
Mr. Malcolm Cecil Foster, Yale University;
Mr. Ronald Martin Foster, American Telephone and Telegraph Company;
Mr. Percy Austin Fraleigh, Cornell University;
Mr. Bennington Pearson Gill, College of the City of New York;
Mr. Rutherford Erwin Gleason, State University of Iowa;
Miss Neoma Lillian Goldsberry, University of Missouri;
Professor Cornelius Gouwens, Iowa State College;
Professor Harry Emil Gudheim, Virginia Polytechnic Institute;
Professor Truman Leigh Hamlin, Clarkson College;
Mr. Robert William Hartley, University of Pennsylvania;
Miss Camilla Hayden, University of Wisconsin;
Dr. Henry Benjamin Hedrick, United States Army Ordnance;
Miss Gertrude Anne Herr, Iowa State College;
Dr. Carl Einar Hille, University of Stockholm;
Professor Allan Wilson Hobbs, University of North Carolina;
Miss Clarice Sarah Hobensack, Ohio State University;
Miss Helma Lou Holmes, University of Texas;
Professor Alfred Hume, University of Mississippi;
Mr. William Anderson Hutcheson, Mutual Life Insurance Company;
Mr. Mark Hoyt Ingraham, University of Wisconsin;
Miss Margaret Eloise Jones, Ohio State University;
Professor Frederick Albert Lewis, University of Alabama;
Dr. Peysah Leyzerah, Lehigh University;
Professor Louis Lindsey, Syracuse University;
Mr. Joe Burton Linker, University of North Carolina;
Mr. Ralph Alden Loring, Dartmouth College;
Mr. Harold Marshall Lufkin, Cornell University;
Professor Edward Hiram McAlister, University of Oregon;
Miss Mary Bell McMillan, Wisconsin State Normal School, River Falls;
Mr. Israel Maizlish, Reed College;
Mr. Edward Lawrence Milne, Lake Forest Academy;
Mr. Thessalon Herbert Milne, University of Alberta;
Mr. Gordon Richmond Mirick, University of Rochester;
Mr. Allen Guy Montgomery, University of West Virginia;
Mr. David Sherman Morse, Cornell University;
Mr. Ray Dickinson Murphy, Equitable Life Assurance Society;
Dr. Almar Naess, Naval Academy of Norway;
Dr. Jason John Nassau, Syracuse University;
Mr. Charles Edward Norwood, United States Army Ordnance;
Professor John Hutcheson Ogburn, Lehigh University;
Professor Earnest Jackson Oglesby, New York University;
Mr. Jesse Otto Osborn, Pennsylvania State College;
Mr. Leroy Elden Peabody, Yale University;
Miss Elsie Marie Plapp, University of Wisconsin;

Mr. Hillel Poritsky, Cornell University;
Miss Jessie Grace Quigley, University of Missouri;
Mr. Ottis Howard Rechard, University of Wisconsin;
Professor Jacob Charles Rietz, Ohio State University;
Professor Lulu Runge, University of Nebraska;
Mr. Howard Conway Shaub, Dartmouth College;
Mr. Charles Robert Sherer, University of Nebraska;
Professor Thomas McNider Simpson, Jr., Randolph-Macon College;
Miss Helen Florene Smith, Iowa State College;
Professor Merlin Grant Smith, Greenville College;
Mr. Marvin Reinhard Solt, Lehigh University;
Professor Charles Stillman Sperry, University of Colorado;
Mr. Eugene Stephens, Washington University;
Mrs. Elizabeth Easton Stickney, University of Illinois;
Miss Louise Elizabeth Cathrine Stuerm, American Telephone and Telegraph Company;
Professor Mary Clegg Suffa, Elmira College;
Mr. Frank Elmer Swift, Notre Dame University ;
Mr. James Henry Taylor, University of Nebraska;
Mr. Van B. Teach, Ohio State University;
Mr. Joseph Miller Thomas, University of Pennsylvania;
Professor Perley Lenwood Thorne, New York University;
Professor Joseph Ellis Trevor, Cornell University;
Professor Joseph Henry Tudor, Pennsylvania State College;
Professor Horace Scudder Uhler, Yale University;
Professor Harry Clark Van Buskirk, California Institute of Technology;
Mr. Charles Conroy Wagner, Pennsylvania State College;
Mr. Lewis Edes Ward, Harvard University;
Professor Emery Ernest Watson, Iowa State College for Teachers;
Professor Warren Weaver, University of Wisconsin;
Professor Ross Albert Wells, Michigan State Normal College;
Mr. Karl Leland Wildes, Massachusetts Institute of Technology;
Professor Robert Daniel Williams, Ohio State University;
Professor Charles Owen Williamson, College of Wooster;
Professor Harold Albert Wilson, Rice Institute;
Mr. Frederick Wood, University of Wisconsin;
Dr. Roscoe Woods, State University of Iowa;
Miss Jessica May Young, Washington University.

Twenty-four applications for membership in the Society were received.

The Council voted to accept the invitation extended by the American Association for the Advancement of Science to affiliate with it. Professor E. B. Van Vleck was appointed a representative of the Society in the division of physical sciences of the National Research Council, as successor to Professor H. S. White, whose term will soon expire.

The final report from Professor E. R. Hedrick, as chairman of the Committee on membership and sales, was received, and the committee was discharged at its own request, with the thanks of the Council for its notable services at this critical time. In all one hundred thirty-two applications for membership and seventy-seven subscriptions to the TRANSACTIONS for the current year have been received through this committee.

Questions of policy were raised concerning dues for foreign members, concerning choice between BULLETIN and TRANSACTIONS in the case of foreign members, concerning sales and exchanges of publications with foreign societies and libraries, and concerning individual or concerted efforts to aid foreign journals. It was voted to authorize the President to appoint a committee to consider these and related questions.

The secretary was requested to write on behalf of the Council a letter of felicitation to Professor Magnus Gösta Mittag-Leffler on the occasion of the seventy-fifth anniversary of his birth, March 16, 1921.

A letter from Professor F. N. Cole was read to the Council, donating to the Society the sum which accompanied the testimonial tendered him at the preceding meeting of the Society in recognition of his very distinguished services. It was voted that the Council accept the gift and extend to Professor Cole its heartiest appreciation of his generosity; it was further voted that this fund shall constitute and be designated as the Cole Fund, and the President was requested to appoint a committee to consider the use to which the income can best be devoted. The Council approved the suggestion that the present volume of the BULLETIN be inscribed to Professor Cole.

The papers read at the February meeting are stated below. Those of Mr. Rice, Professor Glenn, Professor Moore, Dr. Zeldin, and Miss Mullikin, and the second paper of Dr. Ritt, were read by title.

1. Mr. L. H. Rice: *Coefficient of the general term in the expansion of a product of polynomials.*

The writer shows, first, that in the expansion of the product $(a_1 + a_2 + \cdots + a_m)(a_2 + \cdots + a_m)\cdots(a_{m-1} + a_m)a_m$ the coefficient of the general term $Aa_{\alpha_1}^{\beta_1}a_{\alpha_2}^{\beta_2}\cdots a_{\alpha_{f-1}}^{\beta_{f-1}}a_m^{\beta_f}$, where $\alpha_1 < \alpha_2 < \cdots < m$, is:

$$A = \binom{\alpha_1}{\beta_1}\binom{\alpha_2 - \beta_1}{\beta_2}\binom{\alpha_3 - \beta_2 - \beta_1}{\beta_3} \cdots \binom{\alpha_{f-1} - \beta_{f-2} - \cdots - \beta_1}{\beta_{f-1}}.$$

More generally, in the expansion of the product

$$(a_{t_1} + a_{t_1+1} + \cdots + a_n)(a_{t_2} + a_{t_2+1} + \cdots + a_n) \cdots (a_{t_m} + a_{t_m+1} + \cdots + a_n)$$

the coefficient of the general term $Ba_{\alpha_1}^{\beta_1}a_{\alpha_2}^{\beta_2} \cdots a_{\alpha_f}^{\beta_f}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_f$, is

$$B = \binom{\alpha_1'}{\beta_1}\binom{\alpha_2' - \beta_1}{\beta_2}\binom{\alpha_3' - \beta_2 - \beta_1}{\beta_3} \cdots \binom{\alpha'_{f-1} - \beta_{f-2} - \cdots - \beta_1}{\beta_{f-1}},$$

where α_j' is the number of α_j's in the factors of the given product.

2. Professor E. V. Huntington: *The mathematical theory of proportional representation, with a substitute for least squares.*

The author shows that the new method of apportionment which he presented at the December meeting may be based on the following categorical set of postulates (in which A, B, C, $\cdots$ are the populations of the states, and a, b, c, $\cdots$ the number of representatives assigned to each): I. For the case of two states, the ratio between A/a and B/b (or a/A and b/B; or A/B and a/b; or B/A and b/a) should be as near unity as possible. II. No pair of states should be capable of being *improved* by a transfer of representatives within that pair.

3. Professor F. W. Owens: *On the apportionment of representatives.*

This paper compares various used or proposed methods of apportionment, pointing out what might be considered as advantages and disadvantages, with illustrative examples. It is also shown that a justification of the method of major fractions may be found in a form of least squares. Suggestions are also offered as to special treatment of the first representative from each state.

4. Professor Arnold Emch: *On the polar equation of algebraic curves.*

The author studies the transformation

$$(1) \quad \sigma x_1' = x_2(x_3^2 - x_1^2), \quad \sigma x_2' = 2x_1x_2x_3, \quad \sigma x_3' = (x_1^2 + x_3^2)x_3$$

between two planes (x') and (x), which occurs in the change

from polar to cartesian coordinates. If we set $x_1/x_3 = t = \tan(\theta/2)$, $x_2/x_3 = \rho$, $x_1'/x_3' = x$, $x_2'/x_3' = y$, the polar equation of an algebraic curve $F(\rho, \cos\theta, \sin\theta) = 0$ may be written in the form $\phi^*(t, \rho) = 0$, or $\phi(x_1, x_2, x_3) = 0$. By means of the transformation (1) the equation $\phi = 0$ is transformed into the projective equation $f(x_1', x_2', x_3') = 0$ of the algebraic curve $F = 0$. Interpreting ϕ in the projective plane (x), we may speak of the ϕ-curve. The fundamental points of the transformation (1) are the double point $(0, 1, 0)$, and the single points $(1, 0, 0)$, $(1, 0, i)$, $(1, 0, -i)$.

It is shown that to the polar equation in the ϕ-form of order n and deficiency p, in a general position, i.e., independent of the fundamental points, corresponds an algebraic curve f of order $3n$ and the same deficiency. The singularities of f are discussed in detail and are described in full.

Reductions in the order of f and in its multiple points due to the passage of ϕ through the fundamental points or the arrangement of the points of ϕ in couples are also considered. The method outlined in this paper is useful in studying the singularities of an algebraic curve from its polar equation.

5. Professor O. E. Glenn: *Generalization of the concept of invariancy derived from a type of correspondence between functional domains. Second proof of the finiteness of formal binary concomitants modulo p.*

Two functional domains D, Q may be so related that, by a transformation, there exists, for functions φ of D, an invariant relation $\varphi' = r\varphi$ while a correspondence $\varphi \sim \omega$ with functions ω of Q holds which implies also $\omega' = s\omega$. There follows $\varphi = q\omega$ where φ, in some particular cases, is the same function of its arguments as is ω, and in other cases a different function. Combining the above with a theory of invariant elements, the author obtains a proof of the finiteness of formal modular concomitant systems in better alignment with formal methods than his proof given at the December (1920) meeting.

6. Professor R. L. Moore: *Concerning the sum of a countable number of closed point sets.*

In the Tôhoku Mathematical Journal, vol. 13, June, 1918, pp. 300–303, Sierpinski shows that, in a space of m dimensions, if the sum M of a countable number of mutually exclusive closed point sets is closed and bounded, then M is

not connected. He points out that if the stipulation that M be bounded is removed the thus modified proposition holds for $m = 1$ but not for $m = 3$. He raises the question whether it holds for $m = 2$. Professor Moore answers this question by showing the existence of an unbounded closed and connected plane point set which is the sum of a countable number of mutually exclusive closed point sets. He shows, in addition, that Sierpinski's theorem remains true for every m if the stipulation that M be bounded is retained but the stipulation that M be closed is replaced by the stipulation that if there be any limit point of M that does not belong to M then the set of all such limit points be closed.

7. Dr. S. D. Zeldin: *On the simplification of the structure of finite continuous groups with more than one two-parameter invariant subgroup.*

In this paper the author shows how the structure of a group of order $r + 2k$ with $k > 1$ two-parameter invariant subgroups can be simplified if its meroedrically isomorphic group of order r has one invariant spread. He shows that the adjoint of the group G_{r+2k} has $2k + 1$ invariants, one of which is the invariant of the adjoint of the isomorphic group G_r, and by considering the different possible forms of these invariants he finds some of the corresponding structural constants.

8. Dr. J. F. Ritt: *Periodic functions with a multiplication theorem.*

Let $f(z)$ represent a uniform analytic function with no essential singularity in the finite part of the plane, and periodic, either simply or doubly. Dr. Ritt determines all cases in which a number m exists such that $f(mz)$ is a rational function of $f(z)$. The results are practically negative. The modulus of m cannot be less than unity. If $|m| > 1$ and if $f(z)$ is simply periodic, $f(z)$ is a linear function of $e^{\alpha z}$ or of $\cos (\alpha z + \beta)$, where β is restricted to certain values. If $|m| > 1$ and if $f(z)$ is doubly periodic, $f(z)$ must be a linear function of $p(z + \beta)$, except in the lemniscatic and equianharmonic cases. In the lemniscatic case $f(z)$ may be a linear function of $p^2(z + \beta)$ and in the equianharmonic case it may be a linear function either of $p^3(z + \beta)$ or of $p'(z + \beta)$. For $|m| = 1$, there is a wider variety of possibilities.

9. Dr. J. F. Ritt: *Note on equal continuity.*

This paper appears in the present number of this BULLETIN.

10. Professor I. J. Schwatt: *Expressions for the Bernoulli function of order p.*

Eisenlohr, Schlömilch, Worpitzky and others have derived expressions for the Bernoulli function. But their methods assume expansions involving the Bernoulli numbers. The author obtains expressions for the Bernoulli function directly from its definition.

11. Professor I. J. Schwatt: *The expansion of a continued product.*

By means of certain principles of operation with series, the author obtains the expansion of $\Pi_{k=1}^{p} \sin kx$ and also of $\sum_{n=0}^{p} \sum_{k=1}^{n-1} \sin^t (k\pi/n)$ and of similar summations of continued products.

12. Professor I. J. Schwatt: *Method for the summation of a family of series.*

A further extension of the case which the author treated in the NOUVELLES ANNALES DE MATHÉMATIQUES, (4), vol. 16, May, 1916, pp. 203–209, is considered in this paper.

13. Professor I. J. Schwatt: *Note on the evaluation of a definite integral.*

The evaluation of a definite integral from its definition as a summation presents even for some of the simpler cases considerable difficulty. The integrals of expressions which the author is unable to reduce to differentials of known functions lead to series which he cannot sum. Some of the principles involved in the summation of series obtained during the process of evaluating the definite integrals considered in this paper are believed to be new.

14. Mr. John McDonnell: *A property of the Pellian equation with some results derived from it.*

This paper shows that if (t, u) be any solution of the Pellian equation

$$t^2 - Du^2 = 1,$$

then

$$t + u\sqrt{D} = (x\sqrt{M} + y\sqrt{N})^2,$$

where (x, y) is a solution of one of the equations $Mx^2 - Ny^2 = 1$ or $Mx^2 - Ny^2 = 2$, and $MN = D$, where either M or N may be unity. From this result certain conclusions are drawn regarding the fundamental solution of the Pellian equation. It is shown as a further consequence that certain theorems on prime numbers established by Lucas in 1878 may be stated with greater precision, and the criteria derived from them rendered more definite.

15. Miss Anna M. Mullikin (introduced by Professor R. L. Moore): *A necessary and sufficient condition that the sum of two bounded, closed and connected point sets should disconnect the plane.*

The author shows that if, in a space S of two dimensions, m and n are two bounded, closed, and connected point sets, neither of which separates S, then (1) in order that the point set $m + n$ should separate S into at least two parts, it is necessary and sufficient that the set of points common to m and n should not be connected, (2) in order that $m + n$ should separate S into just two connected domains it is sufficient that the set of points common to m and n should be the sum of two closed connected point sets that have no point in common.

16. Dr. T. H. Gronwall: *Some empirical formulas in ballistics.*

The author deduces approximate formulas for the range variations due to a change in temperature and to the rotation of the earth in terms of the variations due to changes in initial velocity, angle of departure and ballistic coefficient, thus making it possible to obtain all the differential variations in range by simple processes of interpolation from the ballistic tables devised by Professor Bennett and now being computed under his direction.

17. Dr. T. H. Gronwall: *Summation of a double series.*

In this paper, it is shown, by the use of Cauchy's theorem, that the double series

$$\sum_{m,n=1}^{\infty} \frac{(m+n-2)!\,(m+n-1)!}{m!\,(m-1)!\,n!\,(n-1)!} x^{2m} y^{2n},$$

which was encountered by Dr. K. W. Lamson in the solution

of a question in mathematical physics, has the sum

$$\tfrac{1}{2}[1 - x^2 - y^2 \\ - \sqrt{(1 + x + y)(1 + x - y)(1 - x + y)(1 - x - y)}],$$

the domain of convergence being $|x| + |y| < 1$.

18. Professor L. P. Eisenhart: *A geometric characterization of the paths of particles in the gravitational field of a mass at rest.*

Schwarzschild integrated the differential equations of a permanent gravitational field in the Einstein theory and obtained the quadratic form

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 + \frac{r}{2m - r} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

The geodesics of the four-space with this linear element are the paths of a particle moving in the gravitational field produced by a mass m at rest at the point $r = 0$. It can be shown that these geodesics can be identified with the geodesics of a three-space Σ with the linear element

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 + \frac{r}{2m - r} dr^2 - r^2 d\theta^2.$$

The author has shown that Σ is characterized by the following properties: (i) it admits a continuous group G_2 of motions into itself; (ii) the ∞^1 surfaces S_1 of Σ which move into themselves for each motion of the group form part of a triply orthogonal system; (iii) the directions of the curves of intersection of the surfaces, S_1, S_2, S_3, of the triple system through a point P are the principal directions of Σ at P, in the sense that one of the surfaces, S_1, S_2, S_3, through P has the maximum gaussian curvature and another the minimum curvature at P of all the surfaces of Σ through P; (iv) the curvature of S_1 is different from zero and equal to the curvature of S_2 or S_3, and the curvature of the other is minus twice that of S_1. This paper will appear in the ANNALS OF MATHEMATICS.

19. Professor A. A. Bennett: *The equations of interior ballistics.*

In this paper, the derivation and interpretation of a set of three fundamental equations for interior ballistics are dis-

cussed. These equations are in approximately the usual form, except for a change of variables. This change of variables is necessitated if tables are to be constructed to give all of the data desired for interior ballistics. The equations have been integrated numerically in the office of the Technical Staff of the Ordnance Department, and tables have been constructed which will shortly be published as a Government Document. The novelty of the undertaking consists in the introduction for the first time of new standard variables, which have a fundamental physical significance. The tables secured are independent of physical dimensions and appear to be the first to be published which correspond directly to the equations in their classical form. The results from the tables must be multiplied by external factors which involve the physical dimensions and depend upon the units chosen.

R. G. D. RICHARDSON,
Secretary.

NOTE ON EQUAL CONTINUITY.

BY DR. J. F. RITT.

(Read before the American Mathematical Society February 26, 1921.)

The notion of equal continuity, introduced by Ascoli, has acquired prominence through applications made of it by Hilbert, Montel and others.* A family of functions, defined on an interval, is equally continuous at a point if for every positive number there is an interval containing the point, in which the oscillation of each of the functions is less than the number.

Dealing with families of functions which are not equally continuous, we shall record here some properties of a function which will be called the *saltus* of the family.

Choosing a point on the interval of definition of the functions, let us suppose that there exists a number for which an interval can be found, containing the point, in which the oscillation of each of the functions is less than the number.

* For the literature on this question, see Montel, *Sur les suites infinies de fonctions*, ANNALES DE L'ECOLE NORMALE, 1907.

The greatest lower bound of all such numbers will be called the *saltus* of the family at the point. If no such number exists, the saltus will be taken as $+\infty$. Of course, wherever the saltus vanishes, the family is equally continuous.

The saltus is an upper semi-continuous function on the interval considered. Consequently, if it vanishes nowhere in an interval, there is a sub-interval within that interval in which its lower bound exceeds zero.*

We shall apply the notion of the saltus of a family of functions to the case of a convergent sequence of continuous functions. Let the sequence of functions

$$\varphi_1(x),\ \varphi_2(x),\ \cdots,\ \varphi_n(x),\ \cdots,$$

each continuous in the interval (a, b),† converge throughout that interval. It is well known that wherever the sequence is equally continuous, the limit of the sequence is continuous.

Suppose that there be an interval (α, β) for no point of which the saltus of the sequence is zero. There must be an interval (α_1, β_1), within (α, β), for every point of which the saltus exceeds some positive number ϵ. Let $\varphi_{i_1}(x)$ be any function of the above sequence, and x_0 any point of (α_1, β_1). In a small neighborhood of x_0, the oscillation of $\varphi_{i_1}(x)$ is very small. But for any such small neighborhood there is a function $\varphi_{i_2}(x)$, with $i_2 > i_1$, whose oscillation in that neighborhood exceeds ϵ. Hence there is an interval (α_2, β_2), interior to (α_1, β_1), in which $\varphi_{i_2}(x)$ differs from $\varphi_{i_1}(x)$ by more than $\epsilon/3$. Continuing thus, we have a sequence of intervals

$$(\alpha_1, \beta_1),\ (\alpha_2, \beta_2),\ \cdots,\ (\alpha_n, \beta_n),\ \cdots,$$

each interior to the preceding one, and a sequence of functions

$$\varphi_{i_1}(x),\ \varphi_{i_2}(x),\ \cdots,\ \varphi_{i_n}(x),\ \cdots,$$

with $i_n > i_{n-1}$, such that

$$|\varphi_{i_n}(x) - \varphi_{i_{n-1}}(x)| > \frac{\epsilon}{3}$$

throughout (α_n, β_n). It is clear that the original sequence could not converge at the points common to the intervals (α_n, β_n).

* Hobson, *Theory of Functions*, p. 238.

† It does not matter whether the interval is taken as open or as closed.

We find thus Baire's well known theorem to the effect that the limit of a sequence of continuous functions is at most point-wise discontinuous.

In connection with convergent sequences of continuous functions, the saltus function here considered can be related with the *measure of non-uniform convergence* introduced by Hobson and Osgood.* These two functions vanish at the same points, which fact shows, of course, that the above proof of Baire's theorem is not fundamentally distinct from that based on the measure of non-uniform convergence. There is no other relation of equality between the two functions.

COLUMBIA UNIVERSITY.

A NEW METHOD IN DIOPHANTINE ANALYSIS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society March 26, 1921.)

1. *Introduction and Summary.* In the preceding number of this BULLETIN (p. 312) I gave reasons why due caution should be observed toward the literature on the solution of homogeneous equations in integers. The valid knowledge concerning this subject is much less than has been usually admitted. The lack of general methods is even greater than in the subject of non-homogeneous equations. The chief aim of the present paper is to suggest such a method, based on the theory of ideals. The method is applicable in simple cases (§§ 2–4) without introducing ideals.

For the sake of brevity we shall restrict attention to the problem of finding all integral solutions of the equation

$$x_1^2 + ax_2^2 + bx_3^2 = x_4^2,$$

an equation admitted† to be difficult of treatment by any known methods, and previously solved completely in integers only in the single case $a = b = 1$.

Let us write

$$x_4 - x_1 = z, \quad x_4 + x_1 = w.$$

Then from the integral solutions of $ax^2 + by^2 = zw$ we must

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* Hobson, loc. cit., p. 484.

† Carmichael, *Diophantine Analysis*, 1915, p. 38.

select those for which $z \equiv w \pmod 2$. The trouble in making such a (simple) selection is avoided when $a = 1$, $b = 4k - 1$, by a reduction* to

$$x^2 + xy + ky^2 = zw.$$

Further, any solution of

$$ax^2 + by^2 = zw$$

yields a solution $X = ax$, $Y = y$, $Z = az$, $W = w$, of $X^2 + abY^2 = ZW$ in which X and Z are divisible by a, and conversely. Hence our problem leads to the canonical equation $N = zw$, where N is $x^2 - my^2$ or $x^2 + xy + ky^2$. We shall see that the theory of algebraic numbers is admirably adapted to the complete solution of $N = zw$ in integers.

2. *Definition of Integral Algebraic Numbers.* Let m be an integer other than 0 and 1, and such that m is not divisible by a perfect square. The numbers $\tau = r + s\sqrt{m}$, where r and s are rational, form a domain of rationality (or field) $R(\sqrt{m})$. Evidently τ and its *conjugate* $\tau' = r - s\sqrt{m}$ are the roots of the equation

$$x^2 - 2rx + r^2 - ms^2 = 0.$$

If the coefficients of this quadratic are all integers, τ and τ' are called *integral algebraic numbers* of $R(\sqrt{m})$. A simple discussion† leads to the following theorem.

THEOREM 1. *The integral algebraic numbers of $R(\sqrt{m})$ are $x + y\theta$, where x and y are integers, and where*

$$\theta = \sqrt{m} \text{ if } m \equiv 2 \text{ or } m \equiv 3 \pmod 4, \tag{1}$$

$$\theta = \tfrac{1}{2}(1 + \sqrt{m}),\ \theta^2 - \theta + \tfrac{1}{4}(1 - m) = 0, \text{ if } m \equiv 1 \pmod 4. \tag{1'}$$

The conjugate to $\xi = x + y\theta$ is $\xi' = x + y\theta'$, where $\theta' = -\sqrt{m}$ in case (1), and $\theta' = \frac{1}{2}(1 - \sqrt{m})$ in case (1'). The product $\xi\xi'$ is called the *norm* of ξ and denoted by $N(\xi)$. Hence, in the respective cases,

* Details are given in the writer's address before the International Mathematical Congress at Strasbourg, in which he described the present method for the simplest cases, including the solution, by use of the arithmetic of quaternions (see PROCEEDINGS LONDON MATH. SOC., 1921), of $x^2 + y^2 + \zeta^2 + \eta^2 = zw$ and hence of $x_1^2 + \cdots + x_5^2 = x_6^2$.

† See this BULLETIN, vol. 13 (1906–7), p. 350, or any text on algebraic numbers.

(2) or (2′) $N(x+y\theta) = x^2 - my^2$ or $x^2 + xy + \frac{1}{4}(1-m)y^2$.

3. *All Rational Solutions of* $N(x+y\theta) = zw$. If $z \neq 0$, we may write

$$\frac{x}{z} = \frac{a}{c}, \qquad \frac{y}{z} = \frac{b}{c},$$

where a, b, and c are integers without a common factor greater than 1. Then we have

$$\frac{w}{z} = \frac{N(x+y\theta)}{z^2} = N\left(\frac{x}{z} + \frac{y\theta}{z}\right) = N\left(\frac{a+b\theta}{c}\right) = \frac{N(a+b\theta)}{c^2}.$$

If we take $\rho = z/c^2$, where ρ is rational, we obtain

(3) $x = \rho ac, \qquad y = \rho bc, \qquad z = \rho c^2, \qquad w = \rho N(a+b\theta)$.

But if $z = 0$, then $N(x+y\theta) = 0$ and $x = y = 0$, and this solution is the case $c = 0$ of (3).

THEOREM 2. *All rational solutions of* $N(x+y\theta) = zw$ *are given by* (3), *where* a, b, c *are integers without a common factor and* ρ *is rational.*

4. *Integral Solutions without the Use of Ideals.* Not all integral solutions of $N(x+y\theta) = zw$ are obtained from (3) by restricting ρ to integral values, as was shown in § 4 of my preceding paper. To obtain further solutions, note that the norm of the product

(4) $$x + y\theta = (a+b\theta)(c+d\theta)$$

of two numbers of our field $R(\theta)$ equals the product of their norms. Hence $N(x+y\theta) = zw$ has the solution

(5) $z = N(c+d\theta), \qquad w = N(a+b\theta), \qquad x, y$ by (4);

or explicitly,

(5_1) $x = ac + mbd, \qquad y = ad + bc, \qquad z = c^2 - md^2,$

$w = a^2 - mb^2, \qquad m \equiv 2$ or $3 \pmod 4$;

(5_2) $x = ac + \frac{(m-1)}{4}bd, \qquad y = ad + bc + bd,$

$z = c^2 + cd + \frac{1}{4}(1-m)d^2, \qquad w = a^2 + ab + \frac{1}{4}(1-m)b^2,$

where $m \equiv 1 \pmod 4$.

We shall restrict attention to integral values of a, b, c, and d.

without a common factor. The products of the resulting numbers (5) by an arbitrary rational number ρ give all rational solutions of $N(x + y\theta) = zw$, since those products reduce to (3) when $d = 0$.

When the field $R(\theta)$ is such that its integral algebraic numbers $x + y\theta$ obey the laws of divisibility of arithmetic, we shall prove that all integral solutions of $N(x + y\theta) = zw$ are given by the products of the numbers (5) by an arbitrary integer ρ, where a, b, c, and d are integers without a common factor. We have merely to show that when the products of the numbers (5) by an irreducible fraction n/p are integers, so that the numbers (5) are divisible by p, then the quotients are expressible in the same form (5) with new integral parameters in place of a, b, c, and d. It suffices to prove this for the prime factors (equal or distinct) of p, since after each of them has been divided out in turn, p itself has been divided out.

Let therefore p be a prime which divides the four numbers (5). If p divided both d and b, it would divide also c and a, in view of z and w, contrary to the hypothesis that a, b, c, and d have no common factor. By the interchange of a with c and b with d, x and y remain unaltered, while z and w are interchanged. Hence we shall be treating one of two entirely similar cases if we assume that d is not divisible by p.

The prime p divides the product z of $c + d\theta$ and $c + d\theta'$, without dividing either factor. For, if $c + d\theta$ or $c + d\theta' = c + d(\alpha - \theta)$, where $\alpha = 0$ or 1 in the respective cases (1) or (1′), were the product of p by $k + l\theta$, where k and l are integers, then $\pm d = pl$, whereas d is not divisible by p. Since the laws of divisibility of arithmetic were assumed to hold for the integral numbers of $R(\theta)$, it follows that p is not an algebraic prime, but decomposes into $p = \pi\pi'$, where π and π' are conjugate primes.* By choice of the notation between π and π', we may assume that π is the one of the two prime factors π and π' of p which divides $c + d\theta$, and we may write

$$(6) \qquad c + d\theta = \pi(C + D\theta), \qquad z = pN(C + D\theta),$$

where C and D are integers. Since x and y are divisible by p,

* Otherwise, p would be a product of three integral algebraic numbers, no one a unit, and its norm p^2 would be a product of three integers no one of which is ± 1. A unit u is an integral algebraic number which divides unity, whence $N(u) = \pm 1$ ($+1$ if m is negative).

and $c + d\theta$ is divisible by π, but not by the product $p = \pi\pi'$, it follows from (4) that $a + b\theta$ is divisible by π', i.e.

$$(7) \qquad a + b\theta = \pi'(A + B\theta), \qquad w = pN(A + B\theta),$$

where A and B are integers.

Comparing the product of (6) and (7) with (4), we have

$$x + y\theta = p(\xi + \eta\theta), \qquad \xi + \eta\theta \equiv (A + B\theta)(C + D\theta).$$

Hence the integral quotients $\xi = x/p$, $\eta = y/p$, z/p, w/p are of the form (5) with a, b, c, and d replaced by the integers A, B, C, and D. This proves the following theorem.

THEOREM 3. *All integral solutions of $N(x + y\theta) = zw$ are obtained by multiplying an arbitrary integer by the numbers (5) in which a, b, c, and d are integers without a common factor (in brief by the formula which expresses the fact that the norm of the product of two numbers $a + b\theta$ and $c + d\theta$ of the field $R(\theta)$ equals the product of their norms), provided the integral algebraic numbers of the field $R(\theta)$ obey the laws of divisibility of arithmetic, and this condition is satisfied only for the following* 45 *values* of m numerically* $\leqq 100$, *m having no square factor*:

$m = -1, -2, -3, -7, -11, -19, -43, -67, 2, 3, 5,$
$6, 7, 11, 13, 14, 17, 19, 21, 22, 23, 29, 33, 37, 38, 41, 43, 46, 47,$
$53, 57, 59, 61, 62, 67, 69, 71, 73, 77, 83, 86, 89, 93, 94, 97.$

5. *Definition of Ideals. Known Theorems.* When the laws of divisibility fail for the integral algebraic numbers of a field, we may restore those laws by the introduction of ideals, as was first done by Kummer for a field defined by a root of unity, and by Dedekind for any algebraic field. By an *ideal* of a field $R(\theta)$ is meant any set S of integral algebraic numbers of $R(\theta)$, not composed of zero only, such that the sum and difference of any two (equal or distinct) numbers of the set S are themselves numbers of this set, while every product of a number of the set S and an integral algebraic number of the field $R(\theta)$ is a number of the set S.

* The cases in which all ideals of the field are principal ideals, J. Sommer, *Zahlentheorie*, 1907, tables, pp. 346–358. Dickson, this BULLETIN, vol. 17 (1910–11), pp. 534–7, proved that if $m = -P$ is negative, 163 is the only value of P between 67 and 1,500,000.

If s ranges over the numbers of an ideal S, and s_1 ranges over the numbers of an ideal S_1, where S and S_1 are ideals of the same field $R(\theta)$, then the products ss_1 and their linear combinations with rational integral coefficients form an ideal of $R(\theta)$, called the *product* of the factors S and S_1, and denoted by SS_1 or by S_1S.

For quadratic fields, the case in which we are here interested, the theory of ideals has been developed quite simply.* Two notations are needed. First, $[k, l]$ denotes the totality of linear homogeneous functions of k and l with rational integral coefficients. Second, $\{k\}$ denotes a *principal ideal*, defined as the totality of the products of k by integral algebraic numbers $x + y\theta$ of our field $R(\theta)$, where x and y are rational integers. Hence we have

$$\{k\} = [k, k\theta].$$

THEOREM 4. *In a quadratic field $R(\theta)$, where θ is defined by* (1) *or* (1′), *all ideals are given by* $[ne, n(f + \theta)]$, *where n, e, f are rational integers such that*

(8) *or* (8′) $\quad f^2 \equiv m \pmod{e}, \quad f^2 + f + \frac{1}{4}(1 - m) \equiv 0 \pmod{e},$

in the respective cases (1) *or* (1′).

THEOREM 5. *In a quadratic field $R(\theta)$, the product of the ideal* $[ne, n(f + \theta)]$ *by its conjugate* $[ne, n(f + \theta')]$ *equals the principal ideal* $\{n^2e\}$.

The positive integer $n^2|e|$ is called the *norm* of $[ne, n(f + \theta)]$. The norm of any principal ideal $\{k\}$ is $|N(k)|$.

An ideal S is said to be *divisible* by an ideal T when there exists an ideal Q of the same field such that $S = TQ$. An ideal, which is different from the principal ideal $\{1\}$ and is divisible by no ideal other than itself and $\{1\}$, is called a *prime ideal.*

THEOREM 6. *If a prime ideal divides AB, it divides A or B. Every ideal which is neither $\{1\}$ nor a prime can be expressed in one and but one way as a product of a finite number of prime ideals. Hence ideals obey the laws of divisibility of arithmetic.*

Two ideals A and B are called *equivalent* if there exist principal ideals $\{\alpha\}$ and $\{\beta\}$ such that $\{\alpha\}A = \{\beta\}B$; we write $A \sim B$. If A is equivalent also to C, with $\{\delta\}A = \{\gamma\}C$,

* Dickson, this BULLETIN, vol. 13 (1906–7), pp. 353–6; and, for a very detailed treatment of the case $m = -5$, ANNALS OF MATH., (2), vol. 18 (1917), pp. 169–178.

then $\{\beta\delta\}B = \{\alpha\gamma\}C$ and B is equivalent to C. Hence all the ideals of a field which are equivalent to a given one are equivalent to each other, and are said to form a *class of ideals*. The *principal class* contains all the principal ideals and no others. For, if $A \sim \{1\}$, $\{\alpha\}A = \{\beta\}$, so that the number β of the product is in $\{\alpha\}$, whence β is divisible by α, and $A = \{\beta/\alpha\}$.

6. *Application of Ideals to our Problem.* We now dispense with the assumption made in § 4 that the integral algebraic numbers of our quadratic field $R(\theta)$ obey the laws of divisibility of arithmetic. We shall examine by means of the theory of ideals our assumption that the solutions (5) of

$$N(x + y\theta) = zw \tag{9}$$

are all divisible by the prime p. The prime ideal $\{p\}$ therefore divides the product $\{z\}$ of the principal ideals $\{c + d\theta\}$ and $\{c + d\theta'\}$, without dividing either of the latter. Hence $\{p\}$ is not a prime ideal (§ 5). Thus $\{p\} = PP'$, where P and P' are conjugate prime ideals (which may coincide). By choice of the notation between P and P', we may assume that P is the one dividing $\{c + d\theta\}$, i.e.

$$\{c + d\theta\} = PL, \tag{10}$$

where L is an ideal of our field $R(\theta)$. By hypothesis, p divides x and y, whence, by (4), $\{p\} = PP'$ divides the product of the principal ideals $\{c + d\theta\}$ and $\{a + b\theta\}$. Since $\{c + d\theta\}$ is divisible by P, but not by $PP' = \{p\}$, it follows that $\{a + b\theta\}$ is divisible by P':

$$\{a + b\theta\} = P'T, \tag{11}$$

where T is an ideal of our field $R(\theta)$.

(a) First, let L be a principal ideal. Then, by (10), P is equivalent to the principal ideal $\{1\}$ and hence P is a principal ideal. Evidently its conjugate P' is a principal ideal. Then, by (11), T is equivalent to, and hence equal to, a principal ideal. Hence equations (10) and (11) between principal ideals yield* equations (6) and (7) between numbers of our field, from which we conclude as in § 4 that the quotients of

* After inserting or removing unit factors, since $\{\pi\} = \{\lambda\}$ implies that λ is the product of π by a unit and conversely.

x, y, z, and w by p are the form (5) with a, b, c, and d replaced by new integers A, B, C, and D.*

(b) Second, let L be equivalent to an ideal S which is not a principal ideal. If S' is the conjugate to S, $SS' = \{e\}$, where e is a rational integer (Theorem 5). Then we shall have $LS' \sim SS' = \{e\}$, so that LS' is a principal ideal $\{\delta\}$. Multiplying (10) by S', we get

$$S'\{c + d\theta\} = P\{\delta\}. \tag{12}$$

Multiplying (12) by S, and using $SS' = \{e\}$, we see that $SP\{\delta\}$ is a principal ideal, so that $SP \sim \{1\}$, whence SP is a principal ideal. Its conjugate $S'P'$ is a principal ideal. But the product of (11) by $SS' = \{e\}$ shows that $S'P' \cdot ST$ is a principal ideal. Thus $ST \sim \{1\}$ and ST is a principal ideal $\{\epsilon\}$. Hence, by (11),

$$S\{a + b\theta\} = P'\{\epsilon\}. \tag{13}$$

By the norms of the members of (12) and (13), in connection with (5), we get

$$|ez| = |eN(c + d\theta)| = p|N(\delta)|,$$
$$|ew| = |eN(a + b\theta)| = p|N(\epsilon)|.$$

By hypothesis, x, y, z, and w are divisible by p. Hence $N(\delta)$ and $N(\epsilon)$ are divisible by e. By comparing the product of (12) and (13) with (4), we get

$$\{e\}\{x + y\theta\} = \{p\}\{\delta\epsilon\}.$$

Hence $e(x + y\theta)/p = \delta\epsilon u$, where u is a unit. Replacing ϵu by ϵ, we conclude that the quotients of x, y, z, and w by p are integers X, Y, Z, and W, such that

$$X + Y\theta = \frac{\delta\epsilon}{e}, \qquad Z = \pm\frac{N(\delta)}{e}, \qquad W = \pm\frac{N(\epsilon)}{e}. \tag{14}$$

The requirement that $\delta\epsilon$, $N(\delta)$, $N(\epsilon)$ be divisible by e may be expressed by congruential conditions modulo e upon the four coordinates of δ and ϵ (cf. § 7). In the future we shall retain only the upper sign in (14) and understand that the

* If we attend only to the numbers z of our solutions, we see that our discussion, with omission of the details leading to (11) and (13), leads to the quadratic forms of all divisors of the numbers represented by the quadratic form N.

simultaneous change of signs of Z and W in any solution leads to a companion solution which will not be listed. Our initial solution (5) is the case $e = 1$ of (14). The removal from (5) of a prime factor led us to the solutions (14). Bearing in mind that we must remove from (5) in succession the various prime factors of the common factor of the numbers (5), we may state the following lemma.

LEMMA. *From each class of ideals of the field $R(\theta)$, where θ is defined in terms of m by* (1) *or* (1′), *select a representative ideal and call its norm e. For each e impose on the coordinates of δ and ϵ the conditions that the divisions indicated in* (14) *are possible and derive the resulting solution* (14) *in integers. Similarly, examine the conditions that the four numbers in any such formula* (14) *shall be divisible by an arbitrary one of the numbers e and derive the resulting solution in integers. Delete one of two such solutions if they are equivalent,* i.e. *if they differ only by a change of integral parameters. Repeat the process until closure results, so that the final sets of solutions $S_1, \cdots, S_k$ are such that, when the numbers of any S_i are divisible by any e, the resulting solution is equivalent to one of $S_1, \cdots, S_k$. Then all integral solutions of $N(x + y\theta) = zw$ are integral multiples of $S_1, \cdots, S_k$.*

The theory of the correspondence* between classes of ideals and classes of quadratic forms and the theory of the composition of classes would seem to entitle us to pass from the preceding result to the following conjectured theorem.

THEOREM 7. *Select a representative $S = [e, f + \theta]$ of each class of ideals of the field $R(\sqrt{m})$; and define θ by* (1) *or* (1′). *Then all integral solutions of $N(x + y\theta) = zw$ are integral multiples of*

$$(15)\quad \begin{aligned} &x = eln + fnq - flr - gqr, && y = lr + nq,\\ &z = el^2 + 2flq + gq^2, && w = en^2 - 2fnr + gr^2,\\ &m \equiv 2 \text{ or } 3 \pmod 4, && f^2 - eg = m, \end{aligned}$$

or of

$$(16)\quad \begin{aligned} &x = eln + fnq - (f+1)lr - gqr, && y = lr + nq,\\ &z = el^2 + (2f+1)lq + gq^2, && w = en^2 - (2f+1)nr + gr^2,\\ &m \equiv 1 \pmod 4, && (2f+1)^2 - 4eg = m. \end{aligned}$$

* To S and its conjugate S' correspond z and w, while to their product (a principal ideal) corresponds $N(x + y\theta)$, which therefore can be obtained from z and w by composition. This suggests another, but more technical, approach to our whole subject.

When S is the principal ideal $\{1\}$, $e = 1, f = 0$, (15), and (16) with n replaced by $n + r$, become (5_1) and (5_2) for $l = c$, $q = d$, $n = a$, $r = b$.

I have verified Theorem 7 for $m = -14, -17, -46$, when there are 4 classes of ideals; for $m = -26$, when there are 6 classes; for all values of m numerically < 100 for which there are 2 or 3 classes (§ 7); and when there is a single class (§ 4). A general proof is being sought by one of my students, who is also applying the method to other types of Diophantine equations.

7. *Cases of* 2 *or* 3 *Classes of Ideals.* The following result, in connection with Theorem 3, disposes of all positive values of $m < 100$, except $m = 82$, in which case alone the number of classes of ideals exceeds 3.

THEOREM 8. *For the* 37 *values of m between* -100 *and* $+100$ *and without a square factor for which there are exactly* 2 *or* 3 *classes of ideals in the field* $R(\sqrt{m})$, *all integral solutions of* $N(x + y\theta) = zw$ *are integral multiples of* (5) *and* (15) *or* (16), *where e and f take the one set or the two sets of values in one or two ideals* $[e, f + \theta]$ *which together with* $\{1\}$ *give representatives of the* 2 *or* 3 *classes of ideals.*

For $m \equiv 2$ or 3 (mod 4), $\theta = \sqrt{m}$, we write

$$\delta = D + q\theta, \qquad \epsilon = E + r\theta. \tag{17}$$

Then

$$\begin{gathered} N(\delta) = D^2 - mq^2, \qquad N(\epsilon) = E^2 - mr^2, \\ \delta\epsilon = DE + mqr + (Dr + Eq)\theta. \end{gathered} \tag{18}$$

First, consider an ideal $S = [e, f + \theta]$ for which e is a prime factor of m. By (8), $f \equiv 0 \pmod{e}$ and we may take $f = 0$. Note that every ideal, not a principal ideal, is equivalent to S when * $m = -6, -10, -22, -58, 10, 26, 30, 42, 58, 70, 74, 78$ with $e = 2$, and when $m = 51$ or 66 with $e = 3$. The numbers (18) are divisible by e if and only if D and E are. For $D = el$, $E = en$, (14) become (15) with $f = 0$.

Second, for $m \equiv 3 \pmod{4}$, let $S = [2, f + \theta]$. By (8), we may take $f = 1$. Note that every ideal, not a principal ideal, is equivalent to S when $m = -5, -13, -37, 15, 35, 39, 55, 87, 91, 95$. Since (18) shall be divisible by 2, $D = q + 2l$, $E = -r + 2n$, and (14) with $e = 2$ become (15).

* Sommer, *Zahlentheorie*, 1907, pp. 346–358. (Tables.)

Third, for $m \equiv 2$ or 3 (mod 4), $m \equiv 1$ (mod 3), let $S = [3, f + \theta]$. By (8), we may take $f = 1$ or -1. For $m = 31$ or 79, the three classes of ideals are represented by $\{1\}$ and the two S's; for $m = 34$, the two classes are represented by $\{1\}$ and S with $f = +1$. The numbers (18) are divisible by 3 if and only if $D \equiv \pm q$, $E \equiv \mp r$ (mod 3). For $D = \pm q + 3l$, $E = \mp r + 3n$, (14) with $e = 3$ become (15) with $f = \pm 1$.

These three main cases cover all the values of m between -100 and $+100$ for which $m \equiv 2$ or 3 (mod 4) and for which there are exactly 2 or 3 classes of ideals.

It remains to verify closure. Let $x, \cdots, w$ in (15) be divisible by e. For our first case, $f = 0$ and $g = -m/e$ is not divisible by e, whence $q \equiv r \equiv 0$ (mod e). The last is true also in our second case $e = 2$, $f = 1$, $g = (1 - m)/2$ odd. For $q = ed$, $r = eb$, the quotients of (15) by e are of the form (5_1) with $a = n - fb$, $c = l + fd$. In the third case $e = 3$, $f = \pm 1$, either $q \equiv r \equiv 0$ (just treated) or $l \equiv fgq$, $n \equiv -fgr$ (mod 3).

First, let $g \equiv -1$ (mod 3) and hence replace l by $-fq + 3l$, and n by $fr + 3n$; the quotients of (15) by 3 are of the form (15) with e replaced by 9, f by $-2f$, and g by $(g + 1)/3$. For $m = 31$, we apply to $z = (9, -2f, -3)$, with $f = \pm 1$, in succession the substitutions

$$\begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix}, \qquad \delta = \pm 2, \mp 1, \pm 1, \mp 5,$$

and obtain $(-3, \pm 4, 5)$, $(5, \pm 1, -6)$, $(-6, \pm 5, 1)$, $(1, 0, -31)$. The product of these substitutions is $l = 2c \mp 11d$, $q = \mp 5c + 28d$. Using also $n = 2a \pm 11b$, $r = \pm 5a + 28b$, we see that our solution becomes (5_1).

Next, let $g \equiv +1$ (mod 3) and hence replace l by $fq + 3l$, and n by $-fr + 3n$; the quotients of (15) by 3 are

$$x = 9ln + 4fnq - 4flr - hqr, \qquad y = lr + nq,$$

$$(19) \qquad z = 9l^2 + 8flq + hq^2, \qquad w = 9n^2 - 8fnr + hr^2,$$

$$h = (g + 5)/3.$$

For $m = 79$, apply to $z = (9, \pm 4, -7)$ in turn

$$\begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix}, \qquad \delta = \pm 1, \qquad \delta = \mp 1.$$

We obtain

$$z = 10s^2 \pm 14st - 3t^2 \equiv \phi \quad \text{for} \quad l = -s \mp t,$$

$q = \mp s - 2t$. But our initial z in (15) with $e = 3, f = \pm 1$, $g = -10$, becomes $-\phi$ for $l = -t \pm 2s$, $q = s$. Eliminating s and t, we see that if we replace l by $\pm l - 3q$, q by $2l \mp 5q$, n by $\mp n - 3q$, and r by $2n \pm 5r$ in (19), we obtain $x, -y, -z, -w$ of our initial (15). But by the change of the signs of q and r in (15), y is changed in sign, while the effect on x, z, w is to change the sign of f.

Finally, let $m = 34$, $e = 3$, $f = +1$. In (19) we take $l = -c + 6d$, $q = c - 5d$, $n = c + 6b$, $r = a + 5b$, and obtain (5_1) with z and w changed in sign.

Let $m \equiv 1 \pmod 4$, $\theta = \frac{1}{2}(1 + \sqrt{m})$, and consider an ideal S of the field $R(\theta)$ for which e is a prime factor of m. By (8′), we may take $2f + 1 = e$. Note that every ideal, not a principal ideal, is equivalent to S when $m = -35, 65, 85$ with $e = 5$; $m = -51$, $e = 3$; and $m = -9$, $e = 7$. By (2′) and (17),

$$N(\delta) = D^2 + Dq + \tfrac{1}{4}(1 - m)q^2,$$

$$N(\epsilon) = E^2 + Er + \tfrac{1}{4}(1 - m)r^2, \tag{20}$$

$$\delta\epsilon = DE + \tfrac{1}{4}(m - 1)qr + (Dr + Eq + rq)\theta.$$

Thus

$$4N(\delta) \equiv (2D + q)^2 \equiv 0, \qquad 4N(\epsilon) \equiv (2E + r)^2 \equiv 0 \pmod e.$$

Hence write $D = fq + el$, $E = -\frac{1}{2}(e + 1)r + en$. Then (14) give (16). To prove closure, let the numbers (16) be divisible by e. Since m is not divisible by e^2, g is not divisible by e. Hence $q \equiv r \equiv 0 \pmod e$. For $q = ed$, $r = eb$, the quotients of (16) by e are of the form (5_2) for $a = n - (f + 1)b$, $c = l + fd$.

Let $m \equiv 1 \pmod{4e}$, where e is a prime. By (8′), we may take $f = 0$ or $f = -1$, and obtain conjugate ideals. The conditions that (20) be divisible by e are

$$D(D + q) \equiv 0, \qquad E(E + r) \equiv 0, \qquad DE \equiv 0,$$

$$(D + q)(E + r) \equiv 0 \pmod e.$$

Hence either $D \equiv E + r \equiv 0$ or $E \equiv D + q \equiv 0$. For $D=el$,

$E = -r + en$, (14) become (16) with $f = 0$. For $E = en$, $D = -q + el$, (14) become (16) with $f = -1$. We may restrict the proof of closure to the first case. For, our two cases are interchanged by the substitution $s = (DE)(qr)(nl)$, which by (17) and (14) gives rise to the interchange of δ with ϵ, and z with w. But s replaces x, y, z, w of (16) with $f = 0$ by x, y, w, z of (16) with $f = -1$. The numbers (16) with $f = 0$ are divisible by e if and only if either $q \equiv r \equiv 0$ (treated above), or $l \equiv -gq$, $n \equiv gr \pmod{e}$.

First, let $g \equiv 0 \pmod{e}$. Replacing l by el and n by en in (16), we see that the quotients by e are of the form (16) with e replaced by e^2 and g by g/e, while f remains zero. For $m = -31$, $e = 2$, whence $g = 4$, we replace l by q, q by $-l$, n by $-r$, and r by n, and obtain (16) with $f = -1$. For $m = -15$, $e = 2$, we replace l by $-d$, q by $c + d$, n by b, and r by $-a$, and obtain (5_2).

Next, let $g \equiv \pm 1 \pmod{e}$. Write $l = \mp q + eQ$, $n = \pm r + eR$. The quotients of (16) with $f = 0$ by e are

$$(21)\quad \begin{aligned} x &= e^2QR - (1 \mp e)rQ \mp eqR - \sigma qr, \quad y = Qr + Rq,\\ z &= e^2Q^2 + (1 \mp 2e)Qq + \sigma q^2, \quad w = e^2R^2 - (1 \mp 2e)Rr + \sigma r^2,\\ \sigma &= 1 + (g \mp 1)/e. \end{aligned}$$

For $m = -23$, $e = 2$, we have $g = 3$ and may choose the upper signs. Replacing Q by $-q$, q by $l - q$, R by r, and r by $-n - r$, we obtain (16) with $f = -1$.

For $e = \sigma = 3$, the upper signs give $g = 7$, $m = -83$, and the lower signs give $g = 5$, $m = -59$. Replacing Q by q, q by $-l \pm q$, R by r, and r by $-n \mp r$, we obtain $-x$, $-y$, z, w of (16) with $f = -1$.

These values -15, -23, -31, -35, -51, -59, -83, -91, 65, 85 are the only ones of m between -100 and $+100$ which are $\equiv 1 \pmod{4}$ and for which there are exactly 2 or 3 classes of ideals. Hence Theorem 8 is proved.

THE UNIVERSITY OF CHICAGO,
February 15, 1920.

$$A(t_1^2 + t_2^2) + Bt_1t_2 + Ct_1t_2(t_1 + t_2) + D(t_1 + t_2) + Et_1^2t_2^2 + F = 0,$$

being necessarily symmetric. The envelope of the line must be of the second class. If $t_{i+p} = t_{i-p}$, the relation must be doubly linear and the envelope must be of the first class. This would occur if $n + 1 = 2m$ and $p = m$.

The polynomial whose roots are $t_1, \cdots, t_{n+1}$, as has been seen, must be of the form $f(t) + \lambda\varphi(t)$, or must belong to an involution. The double elements of the involution are $2n$ in number and are the parameters of points in which two vertices of a polygon coincide. These polygons are found by starting from a common point of s and Σ or from a point of contact of a common tangent. If $n + 1$ is odd, starting from a common point the polygon must consist of a succession of segments counted twice, and the tangent at the extremity, which must be a tangent of Σ. There are four such polygons, and the corresponding forms of the involution must be

$$f + \lambda_i\varphi = (t - \alpha_i)\psi_i^2 \qquad (i = 1, 2, 3, 4)$$

where α_i is the parameter of an intersection. If $n + 1$ is even, the singular polygons connect intersections or connect contact points of common tangents, and the forms of the involution are

$$f + \lambda_1\varphi = (t - \alpha_1)(t - \alpha_2)\psi_1^2,\ f + \lambda_3\varphi = (t - \beta_1^2)(t - \beta_2^2)\psi_3^2,$$

$$f + \lambda_2\varphi = (t - \alpha_3)(t - \alpha_4)\psi_2^2,\ f + \lambda_4\varphi = (t - \beta_3^2)(t - \beta_4)^2\psi_4^2,$$

where β_1, β_2, β_3 and β_4 are the parameters of the points of contact. Since the double elements are properly accounted for, there are only four branch forms.

The forms f, φ are seen to be subject to the same conditions as are the forms in Jacobi's problem. Conversely, if the involution $f + \lambda\varphi$ possesses branch forms of the same character as those that are required in Jacobi's problem, the forms f and φ lead to two conics in the poristic relation, and the involution curve decomposes. Suppose $n + 1$ odd and equal to $2m + 1$. Then to a polygon $(t - \alpha)\psi^2$ correspond $m(m - 1)/2$ double tangents of the involution curve, or $2m(m - 1)$ for the four forms. A curve of class $2m$ can only have this number if it decomposes. Every addition to the maximum number of double tangents for a proper curve must

be due to a further decomposition. For we have

$$1 + \frac{(p+q-1)(p+q-2)}{1}$$
$$= \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + pq,$$

and since

$$2m(m-1) = \frac{(2m-1)(2m-2)}{2} + m - 1,$$

the involution curve must consist of m parts. In the special case where the curve is an involution curve these must all be conics. For let $p_1, \cdots, p_m$ be the classes of the components, so that $p_1 + \cdots + p_m = 2m$; then some class must be equal to 2 or to 1. But if a point p forms part of the involution curve, the order of the involution is even, since points on s collinear with P belong simultaneously to the involution. The class of some part must be 2 and a conic Σ must be inscribed in lines of the system. If Σ is tangent to fewer than $n+1$ lines, the forms f and φ must have a common factor, which is excluded by the assumption of a proper solution of Jacobi's problem. Then Σ touches $n+1$ lines, and by the theory given above the involution curve completely decomposes into conics. If $n+1$ is even the conclusion is similar: the involution curve consists of conics and one point.

3. *Closure.* To effect the solution of Jacobi's system, it is necessary to consider the condition for closure. This is known under various forms. It is convenient to use a recurrence formula. Let there be two conics referred to the common self-polar triangle (A) $x^2 + y^2 + z^2 = 0$, (B) $ax^2 + by^2 + cz^2 = 0$. Take M_0 a point on A and let D_1, the polar of M_0 with respect to B, meet A in M_1, M_{-1}. The polar of M_1 meets A in M_0 and a point M_2 and the polar of M_{-1} meets A in M_0 and M_{-2}. Let D_2 be the line M_2M_{-2}. In this way may be derived a series of points M_pM_{-p} and lines D_p; the envelope of D_p may be called A_p. From these definitions it follows, letting M_0 be $\xi\eta\zeta$, that the equation of D_0 is $\xi x + \eta y + \zeta\tau = 0$, that of D_1 is $a\xi x + b\eta y + c\zeta z = 0$, and the equation of D_n is $a_n\xi x + b_n\eta y + c_n\zeta z = 0$, where a_n, b_n, and c_n are functions of a, b, and c independent of $\xi\eta\zeta$ and determined by the following relations of recurrence:

$$a_0 = b_0 = c_0 = 1, \qquad a_1 = a, \qquad b_1 = b, \qquad c_1 = c,$$
$$b_{2p} + c_{2p} = 2b_p{}^2c_p{}^2, \qquad cb_{2p-1} + bc_{2p-1} = 2b_{p-1}c_{p-1}b_pc_p,$$
$$c_{2p} + a_{2p} = 2c_p{}^2a_p{}^2, \qquad ac_{2p-1} + ca_{2p-1} = 2c_{p-1}a_{p-1}c_pa_p,$$
$$a_{2p} + b_{2p} = 2a_p{}^2b_p{}^2, \qquad ba_{2p-1} + ab_{2p-1} = 2a_{p-1}b_{p-1}a_pb_p.$$

There results the system of equations

$$\frac{b_n{}^2 - c_n{}^2}{b^2 - c^2} = \frac{c_n{}^2 - a_n{}^2}{c^2 - a^2} = \frac{a_n{}^2 - b_n{}^2}{a^2 - b^2},$$

$$\frac{b^2c_n{}^2 - c^2b_n{}^2}{b^2 - c^2} = \frac{c^2a_n{}^2 - a^2c_n{}^2}{c^2 - a^2} = \frac{a^2b_n{}^2 - b^2a_n{}^2}{a^2 - b^2},$$

from which follow

$$a_n{}^2 = G_na^2 + H_n, \qquad b_n{}^2 = G_nb^2 + H_n, \qquad c_n{}^2 = G_nc^2 + H_n,$$

where G_n and H_n are symmetric functions of a^2, b^2, and c^2, with $G_0 = 0$, $H_0 = 1$, $G_1 = 1$, $H_1 = 0$. They satisfy the equations

$$G_{p-1}G_{p+1} = H_p{}^2, \qquad G_{2p+1} = (G_pH_{p+1} - G_{p+1}H_p)^2,$$
$$G_{2p} = 4a_p{}^2b_p{}^2c_p{}^2G_p, \qquad G_2 = 4a^2b^2c^2.$$

Hence G_n is the square of a symmetric function of degree $n^2 - 1$ in a, b, and c; i.e. $G_n = \Lambda_n{}^2$. The sign of Λ_n is determined by the equations $\Lambda_2 = 2abc$, $\Lambda_{p-1}\Lambda_{p+1} = -H_p$.

The envelope of D_n is

$$a_n{}^2x^2 + b_n{}^2y^2 + c_n{}^2z^2,$$

or

$$\Lambda_n{}^2(a^2x^2 + b^2y^2 + c^2z^2) - \Lambda_{n-1}\Lambda_{n+1}(x^2 + y^2 + z^2) = 0.$$

If A_n coincides with A_0 the condition is $\Lambda_n = 0$. The condition that a polygon of n sides inscribed in the conic $x^2 + y^2 + z^2 = 0$ is circumscribed about $a^2x^2 + b^2y^2 + c^2z^2 = 0$ is $\Lambda_n = 0$.

Involutions of the required character may be constructed from the equations of the lines D_p. Taking the cubic involution, we find that the condition for triangular closure is

$$(bc+ca+ab)(-bc+ca+ab)(bc-ca+ab)(bc+ca-ab)=0$$

or

$$b_2c_2 + c_2a_2 + a_2b_2 = 0.$$

If we set $\xi = 2t_0$, $\eta = 1 - t_0^2$, $\zeta = i(1 + t_0^2)$, the equation $f = 0$, where

$$f = (t - t_0)\{4a_2t_0t + b_2(1 - t_0^2)(1 - t^2) - c_2(1 + t_0^2)(1 + t^2)\},$$

gives the parameters of the points M_0, M_2, and M_{-2}. If we suppose $b_2c_2 + c_2a_2 + a_2b_2 = 0$, and let t_0 vary, the forms of a cubic involution are determined. In fact, the factor

$$4a_2t_0t + b_2(1 - t_0^2)(1 - t^2) - c_2(1 + t_0^2)(1 + t^2)$$

gives the affinity equation of the involution when equated to zero. The double elements are found by putting $t_0 = t$ and they are the roots of the equation

$$4a_2t^2 + b_2(1 - t^2)^2 - c_2(1 + t^2)^2 = 0.$$

Let us call these roots δ_1, δ_2, δ_3, and δ_4. The branch elements ϵ_1, ϵ_2, ϵ_3, and ϵ_4 are the values of t_0 such that the affinity quadratic is a square. Hence they must satisfy the equation

$$(c_2^2 - b_2^2)t^4 + 2(a_2^2 + b_2^2 - 2a_2^2)t^2 + c_2^2 - b_2^2 = 0.$$

The equations for δ and ϵ are reciprocal equations involving only even powers of t, and it may easily be found that the branch forms of the involution are of the form

$$\lambda f+\mu\varphi=A_1(x+\epsilon)(x+\delta)^2, \qquad \mu f+\lambda\varphi=A_3\left(x+\frac{1}{\epsilon}\right)\left(x+\frac{1}{\delta}\right)^2,$$

$$\lambda f-\mu\varphi=A_2(x-\epsilon)(x-\delta)^2, \qquad \mu f+\lambda\varphi=A_4\left(x-\frac{1}{\epsilon}\right)\left(x-\frac{1}{\delta}\right)^2,$$

where δ is a root of the equation for the double elements and ϵ is the corresponding value of the branch element. It is easy to complete the solution of the transformation problem. It is seen that Cayley's normal form of the elliptic integral appears here.

If the order is 4, the involution is given by the equation

$$f = \{4at_0t + b(1 - t_0^2)(1 - t^2) - c(1 + t_0^2)(1 + t^2)\}\{4a_3t_0t + b_3(1 - t_0^2)(1 + t^2) - c_3(1 + t_0^2)(1 + t^2)\},$$

with $\Lambda_4 = 4abca_2b_2c_2 = 0$, the roots of $f = 0$ being the parameters of the points M_1, M_{-1}, M_3, and M_{-3}. Suppose $a_2 = 0$, then $a_3 = - ab_2c_2$, $b_3 = bb_2c_2$, and $c_3 = cb_2c_2$; hence,

omitting the factor b_2c_2, the involution consists of the forms

$$b^2(1-t_0^2)^2(1-t^2)^2+c^2(1+t_0^2)^2(1+t^2)^2 - 2bc(1-t_0^4)(1-t^4)-16a^2t_0^2t^2,$$

with $a_2 = -b^2c^2+c^2a^2+a^2b^2=0$, or

$$[(c-b)t_0^2+c+b]^2t^4+[(c+b)t_0^2+c-b]^2 + 2\frac{c^2-b^2}{c^2+b^2}[(c^2+b^2)t_0^4+2(c^2-b^2)t_0^2+c^2+b^2]t^2,$$

or, if we set $[(c-b)t_0^2+c+b]^2=\lambda$ and $[(c+b)t_0^2+c-b]^2=\mu$, of the forms

$$\lambda t^4+\frac{c^2-b^2}{c^2+b^2}(\lambda+\mu)t^2+\mu.$$

For the fifth order the affinity equation is

$$[4a_2t_0t+b_2(1-t_0^2)(1-t^2)-c_2(1+t_0^2)(1+t^2)] \times[4a_4t_0t-b_4(1-t_0^2)(1-t^2)-c_4(1+t_0^2)(1+t^2)]$$

with $\Lambda_5=0$. If we set $p=b_2c_2$, $q=c_2a_2$ and $r=a_2b_2$ it is found that

$$pqr\cdot\Lambda_5=(pq+qr-rp)(qr+rp-pq)(rp+pq-qr) - pqr(p+q+r)^3$$

and

$$a_4=q^2+r^2-p^2,\quad b_4=r^2+p^2-q^2,\quad c_4=p^2+q^2-r^2.$$

The discriminant of the involution is found by putting $t_0=t$ in the affinity equation. It must also be given by the resultant of the two factors, since a, b, and c are subject to the relation $\Lambda_5=0$. If we develop the two forms of the discriminant and compare the coefficients, a number of forms of the closure condition result, but there are extraneous factors.

This method of constructing involutions is general and furnishes a complete solution of Jacobi's problem. The problem is always possible and determinate. It has been assumed that a, b, and c are subject to no other relation than $\Lambda_n=0$. Such cases are of importance in the theory. Since the present purpose is to solve Jacobi's problem in its general form, they are left for further consideration.

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BACHMANN ON FERMAT'S LAST THEOREM.

Das Fermatproblem in seiner bisherigen Entwickelung. By Paul Bachmann. Berlin and Leipzig, Walter de Gruyter, 1919. pp. viii + 160.

This volume reproduces to a considerable extent most of the important contributions which have so far been made toward a proof of Fermat's last theorem. It is far more complete than anything of the sort heretofore published. In particular, a reader of the book will find therein an account of the main results of Kummer, with proofs in most cases set forth in full. The writer wishes to call attention to the fact, however, that a number of references to articles bearing directly on some of the work given in the text have been omitted by Bachmann, a few of which will be noted, in detail, presently. If a better historical perspective is desired, it would be well for a reader to examine at the same time chapter 26, volume 2, of Dickson's *History of the Theory of Numbers.*

I shall now point out some parts of the text which give an account of results not given in detail elsewhere, aside from the original articles.* Consider

$$x^p + y^p + z^p = 0, \tag{1}$$

where x, y and z are rational integers, prime to each other, and p is an odd prime. The assumption that xyz is prime to p

* For an account of the more elementary results regarding the theorem, cf. Carmichael, *Diophantine Analysis*, chap. 5, or Bachmann, *Niedere Zahlentheorie*, vol. 2, chap. 9.

will be referred to as case I and the contrary assumption, $xyz \equiv 0 \pmod{p}$, as case II. In §§ 14–15 the researches of A. Fleck* are given. In §§ 23–24 the principles underlying Dickson's extensions of the method of Sophie Germain are set forth and the former's result is given that (1) is not satisfied in Case I for any $p < 7{,}000$ except perhaps 6,857.†

The proof of Dickson's theorem that there is at least one set of solutions in integers x, y and z, prime to each other and the prime q, of

$$x^p + y^p + z^p \equiv 0 \pmod{q},$$

if

$$q \geqq (p-1)^2(p-2)^2 + 6p - 2,$$

is reproduced in full (§§ 25–26) as well as the proof by Schur of an analogous result.

In § 32, Bachmann begins the treatment of (1) by the use of the cyclotomic field theory following the methods of Kummer. The class number h of the field defined by $e^{2i\pi/p}$ is referred to and the following statement (p. 104) is made:

"Die Kummersche Untersuchung ergibt ferner dass die Klassenzahl des Kreisteilungskörpers nur einmal durch eine Primzahl p teilbar ist, wenn diese nur in einer der gennanten Bernoullischen Zahlen aufgeht . . . ."

The class number h may have this property but Kummer's work does not prove it.‡

In §§ 36–37 is given substantially Hilbert's form of Kummer's proof that

$$\alpha^p + \beta^p + \gamma^p = 0$$

has no solution in integers α, β, γ, belonging to the cyclotomic field defined by $e^{2i\pi/p}$, provided p is a *regular* prime.

On page 111, the results given in Kummer's memoir§ of 1857 on Fermat's last theorem are mentioned, but no part of the argument is reproduced, except the derivation of the so-called Kummer criteria, namely that if (1) is satisfied in

* Sitzungsberichte Math. Gesellschaft Berlin, vol. 8, p. 133, and vol. 9, p. 50.

† Dickson states that he has proved the result also for $p = 6857$, but he has not published the details. This would also follow from the relation $7 \times 6857 = 3 \times 2^7 \times 5^3 - 1$. See Vandiver, Trans. Amer. Math. Soc., vol. 15, p. 204.

‡ See the writer's criticism of a similar statement by Kummer, Proceedings Nat. Acad. Sciences, vol. 6, p. 266 (May, 1920).

§ Abhandlungen, Berlin Academy, 1857, pp. 41–74.

Case I, then the congruences

$$(2) \qquad B_s \frac{d_0{}^{p-2s} \log (x + e^v y)}{dv^{p-2s}} \equiv 0 \pmod{p} \qquad (s = 1, 2, \cdots, (p-3)/2),$$

all hold, where the B's are the Bernoulli numbers, $B_1 = 1/6$, $B_2 = 1/30$, etc., and where the symbol $d_0{}^{p-2s}/dv^{p-2s}$ means that zero is substituted for v after the differentiation is performed. Setting $x + e^v y = (x, y)$, then the same congruences hold with (y, x), (x, z), (z, x), (y, z) and (z, y) substituted for (x, y). The greater part of Kummer's proof of (2) is reproduced, but there are several points in the work which are not brought out by either Kummer or Bachmann, and which might puzzle a reader going over it for the first time. For example our author does not reproduce the argument employed by Kummer to establish the relation

$$(3) \qquad \Pi_i (x + \alpha^{g^i} y) = \pm \alpha^m f(\alpha)^p,$$

where g is a primitive root of p, and where i ranges over the integers in the set $0, 1, 2, \cdots, p-2$, which have the property $g_{\mu-1} + g_{\mu-1+\text{ind}\, r} > p$, where $\mu = (p-1)/2$ and g_s is defined as the least positive residue of g^s, modulo p, $r = 1, 2, \cdots, p-2$, $g^{\text{ind}\, r} \equiv r \pmod{p}$ and $\alpha = e^{2i\pi/p}$. In this connection it may be noted that the writer has not been able to justify Kummer's method* of showing that $\pm \alpha^m$ is the particular type of unit which appears in this relation.†

More details in the derivation of some of the other results regarding (3) would have been distinctly helpful to the reader; for example, in the derivation of relation 159 on page 114, and of the relation at the bottom of page 115.

On page 123, Bachmann outlines the method of Mirimanoff for proving that

$$\varphi(t) = (1+t)^{p-i} P_i(1, t) \qquad (i = 2, 3, \cdots, p-1)$$
$$\equiv t - 2^{i-1} t^2 + 3^{i-1} t^3 - \cdots - (p-1)^{i-1} t^{p-1} \pmod{p}$$

where

$$\frac{d_0{}^i \log (x + e^v y)}{dv^i} = \frac{P_i(x, y)}{(x+y)^i}.$$

* Kummer, loc. cit., p. 62, and CRELLE, vol. 35, p. 364.

† Compare with the writer's treatment of a similar problem, ANNALS OF MATHEMATICS, vol. 21, No. 2, Dec., 1919, pp. 74–75.

No reference is made to the much simpler method due to Frobenius* for obtaining the same result.

In connection with the derivation of the Kummer criteria, Bachmann does not mention (except on page 126 as to a minor detail) the researches of Cauchy, who anticipated Kummer in obtaining some of the important results related to these criteria. Cauchy† gave without proof a relation equivalent to (3) of this review for $r = 1$, and also stated that if (1) is possible in case I, then

$$1^{p-4} + 2^{p-4} + \cdots + ((p-1)/2)^{p-4} \equiv 0 \pmod{p},$$

which is a transformation of the criterion $B_{\mu-1} \equiv 0 \pmod{p}$. A consideration of these results in connection with the fact that Cauchy gave‡ a theorem regarding functions having properties similar to those of $\varphi(t)$, indicates the probability that he had obtained relations equivalent to two or more of the criteria of Kummer.

In §§ 47–48 an account is given of Frobenius' derivation§ of Mirimanoff's transformation of (2) which led the latter to the criteria $2^{p-1} \equiv 3^{p-1} \equiv 1 \pmod{p^2}$, for the solution of (1) in case I. This work of Frobenius is based on the symbolic method of Blissard|| for treating formulas involving Bernoulli's numbers.

On page 150, Bachmann makes a statement which would lead a reader to suppose that this work of Frobenius should be regarded as an introduction to the latter's later paper on Fermat's last theorem, the contents of which are not given by Bachmann. This is misleading, as the method employed in the second paper for deriving the relation

$$(x^p - 1)G_m{}^k(x) - (x^m - 1)F_{m:k}(x) \equiv H_m{}^{(k)}(x) \pmod{p}$$

is based on an extension of the method used by the writer¶ and is quite different from the method of Frobenius' earlier paper.

The book will constitute a valuable aid to anyone attempting a serious study of Fermat's last theorem.

H. S. VANDIVER.

* SITZUNGSBERICHTE MATH. GESELLSCHAFT BERLIN, July, 1910, p. 843.
† *Oeuvres*, (1), vol. 10, p. 362, Th. 3, and p. 364, Cor. 2.
‡ Loc. cit., p. 356, Th. 5.
§ SITZUNGSBERICHTE MATH. GESELLSCHAFT BERLIN, 1910, p. 200.
|| QUARTERLY JOURNAL, vol. 4, 1861, p. 279. Erroneously attributed to Lucas by numerous writers.
¶ CRELLE, vol. 144, p. 314.

TWO BOOKS ABOUT AIRPLANES.

Aeronautics: A class text. By Edwin B. Wilson. New York. John Wiley and Sons. vi + 265 pp.

Grundlagen der Flugtechnik: Entwerfen und Berechnung von Flugzeugen. Von Dr.-Ing. H. G. Bader. Berlin, B. G. Teubner, 1920. Mit 47 Figuren im Text. vi + 194 pp.

The mathematical treatment of the motion of an airplane requires a knowledge of two subjects, the dynamics of a rigid body, and the theory of fluid motion. The particular application is that of a rigid body moving in any way through a gas and supported by the reaction of the gas. In order to get this support, it is necessary that the body should have such a shape that the pressure of the gas may have full play, and that there be attached a power system which, by driving the body through the air, shall furnish a support equal to the weight of the body. If the body is required to rise, additional power is necessary to furnish an upward force to create the vertical acceleration. The first problem is, therefore, a consideration of the pressures to be produced when the body is to have uniform motion in a horizontal straight line, and next, the additional pressures when it is required to rise. Moreover, the best shape of the body for the purpose in view must be considered.

It is a familiar observation in watching the flight through the air of any object, such as a thin plate or a sheet of cardboard, that it rarely keeps in a fixed direction, or remains parallel to its first position, unless some device is used for the purpose. Even then there will be considerable deviations. In fact, one may almost say that, in general, the motion of a body through a fluid is unstable, and that, in order to keep it steady on a straight track, it is necessary to check the slight deviations as they arise. In other words, human agency is required for steering such an object. On the other hand, careful research, both mathematical and practical, have shown that we can often find devices which will maintain stability within certain limits. In the case of the airplane, investigation of the stability of any particular type is of fundamental importance, and this should extend not only to small deviations, but also to any that are likely to arise.

As far as the motion of the body is concerned when the various forces are given, the well known methods and results of ordinary dynamics can be applied immediately with full confidence. There are no fundamental difficulties to be overcome, except that of reducing complicated formulas to arithmetic. This is chiefly a matter of calculation, which is often laborious. The forces are of three kinds, those due to gravitation which can always be calculated, those due to the thrust of the power system for which we usually have the necessary data, and finally, those due to the air pressures. Theoretically, the latter should be deducible from the equations of hydrodynamics, but practically so little has been achieved in this effort, that in most cases recourse has been had to experiment both for the general laws and the detailed results.

It is really rather astonishing how little has been deduced for even the simplest case, say that of a sphere moving with constant velocity through a liquid. Solutions are known and are fully worked out for the case of a fluid without viscosity. But if the velocity exceeds a certain rather low limit, and the fluid has only a slight viscosity, the motion of the fluid and its pressures on the surface of the sphere bear little or no resemblance to those given by the solution for a non-viscous fluid. The motion is said to beome *turbulent*, which is merely a word to express our ignorance of its essential properties and our inability to calculate it. The mathematical solution for a perfect fluid cannot be treated as a first approximation to that in a slightly viscous fluid. The same fact vitiates nearly all the cases hitherto treated. This is made particularly obvious when we open the chapter on viscosity in a standard treatise, such as Lamb's *Hydrodynamics*, and find that in the very simplest problems the coefficient of viscosity appears in the denominator so that its vanishing makes the expressions infinite. Of course, there is an explanation of this result, but the explanation is after all really an apology for the inherent weakness of all the known methods of treatment for the motion of a solid through a liquid.

Thus, when we come to learn about the distribution of air pressures on a moving plane, we find that hydrodynamics helps but little. This is partly due to circumstances. During the war, when the greatest advance was made in the development of flying machines, it was necessary to obtain results as quickly as possible, and as there was little hope of getting them from

theory, experimental data were gathered by a numerous body of workers. It is from them that most of our present knowledge comes. But with calmer times, it is to be hoped that theory may take its proper share in the investigations and not only gather up into general formulas the known* numerical data, but also deduce further consequences and make advances on its own account.

Two or three volumes on the mathematics of the airplane have appeared in England. So far as we are aware, Professor Wilson's is the first to appear in this country. He is unusually well qualified for the task. Besides his eminence as a mathematician and a physicist, he has taught the subject for some years at the Massachusetts Institute of Technology, and has himself contributed a valuable memoir. Thus his views as to the proper approach for the student merit careful consideration. Like most of those who write introductory textbooks (and his is frankly of this character), he is naturally influenced by the character and degree of knowledge of the students who attend his own classes. This may cause difference of opinion as to what portions of previous subjects should be included. Aside from this fact, however, one can see well what is needed for the student approaching the subject of aeronautics, and the teacher can easily omit or add such matter as he may find necessary.

The first three chapters of Professor Wilson's text, constituting an introduction, give the principal facts which the student needs to know about the motion of a plane through the air. In the second chapter, the accepted law of pressure, —that it varies as the square of the velocity, the area of the surface, and the sine of the angle of attack (the angle between the plane and its direction of motion relative to the air),— is discussed, and the deviations from this statement for different velocities, shapes, and angles, as shown by experiment, are given and are illustrated by diagrams. With this as a basis, the equations of relative equilibrium of the skeleton plane are obtained in the third chapter, for steady horizontal, inclined, and circular flight. The effect of a second plane attached to the first as a *stabiliser* is also treated.

The next five chapters under the sub-title *Rigid Mechanics*, deal with the motion of the airplane under various conditions.

* The word *known* is perhaps inapplicable, for much of the information gathered by the various governmental stations is not yet available.

In the first of them, entitled *Motion in a resisting medium*, the landing problem, a free vertical fall and some other trajectories are discussed. In connection with the free fall, Professor Wilson uses an illustration showing how to construct an analytic expression which shall fit a table of numerical results derived from experiment. It may interest some readers to know that there *is* a maximum velocity of descent. Next follows a discussion of harmonic motion, which is not only sufficient for the purpose in view, but can be recommended as an excellent introduction to the subject apart from its present applications. Then follow three chapters which constitute the main treatment of the motion of an airplane from the point of view of a rigid body. The method is to form the general equations of motion, assume some form of steady motion which satisfies them, and then to determine the stability by finding the character of the small deviations from this steady motion, that is whether they are oscillatory, damped, or increase with the time. Professor Wilson introduces the student to the nature of the problem by solving some cases in two dimensions first and treating the general case in a later chapter. Those who have attempted to present Bryan's excellent but somewhat involved treatment of this subject to a class will welcome these chapters, which give the main portion of the work in sufficient detail for a good grasp of the methods to be obtained.

The third section, entitled *Fluid Mechanics*, deals partly with theoretical hydrodynamics, and partly with experimental results of observations in wind-tunnels and elsewhere. The author has tried to make the best of a bad job, but we fear that the student will recognize rather easily how little the theory assists in giving information about the air forces on an airplane. It is none the less necessary that he should learn at least the elements of fluid motion from the theoretical side, for one cannot doubt that the time will come when they will get fuller application.

A chapter on dimensions, besides having great importance for aerodynamics, on account of the necessary study of the behavior of models, fills a gap. Its use in determining the form of a functional relation between physical quantities when we know what variables are present, is occasionally hinted at but is not often adequately treated in a text-book.

Throughout the whole volume, numerical examples are used

as illustrations, most of them being taken from existing machines. In addition, Professor Wilson has gathered together at the end of each chapter numerous problems for the student to solve. If we regret anything, it is the lack of references to further work for the interested student or at least to show the rank and file they have not reached the conclusion of all knowledge on the subject. But this is a mere detail which will be filled in by the instructor who uses the book.

As its title denotes, the volume by Dr. Bader is devoted to the technique of the construction of airplanes, and as such it has no very special interest to the mathematician. But it is worth noting that the author is obviously well acquainted with the applications of mathematics to dynamical problems, both from the evidence of the book itself and from the titles of his own papers given in the list of the forty-three references at the end. He is full of his subject and he realizes that, in spite of basing all his results on theory, after all the final test must come from experiment. The maze of symbols used and the references to other work in the shape of formulas taken bodily with little explanation, repels the reader to some extent. If one searches for information as to the progress made in Germany in airplane construction, he will not find very much disclosed. Nevertheless, the aerodynamic expert need not consider the volume a useless addition to his library, for there are numerous hints here and there which will well repay some thought and examination. The copy received by the reviewer in paper covers fell to pieces on opening and is now a bundle of loose sheets. The price quoted on a slip enclosed with it is $1.30, which at the present rate of exchange, is equivalent to about 80 marks. This is probably about five times the price charged in the home market.

E. W. Brown.

SHORTER NOTICES.

Die Quadratur des Kreises. By Eugen Beutel. Leipzig and Berlin, Teubner, 1920. 56 pp.

After stating clearly what is meant by a ruler and compass construction, the author gives a very complete history of this famous problem, which baffled mathematicians for twenty-five hundred years. He divides the history into three periods, the *geometric* period, the period of infinite series, the algebraic period. The first era, in which men tried to solve the problem by purely geometric methods, includes the work done by the Egyptians, Babylonians, Greeks, Romans, Hindus, Chinese, and the Christian nations up to the time of Newton. The work of Archimedes and of van Ceulen are the outstanding contributions in this period. When we stop to think that Archimedes was unfamiliar with the Arabic numerals and the decimal notation, we appreciate the magnitude of his task of computing the perimeter of the inscribed and circumscribed polygon of 96-sides in a circle of radius R. He found $3\frac{1}{7} < \pi < 3\frac{10}{71}$. Ludolf van Ceulen (1539–1610) carried Archimedes process of approximation still further and obtained the value of π correct to 35 decimal places. It is in honor of this contribution that the Germans today often refer to π as the Ludolfian number.

The second period contains the work of such men as Newton, Leibnitz, Gregorius von St. Vincentius, Kepler, Fermat and John Wallis. By means of infinite series these men were able to compute the value of π to several hundred decimal places, but they were unable to solve the problem. However, their investigations led them to believe that the problem was incapable of solution.

It was then shown that if the problem is solvable, π must not be transcendental. In 1873 Hermite proved that e was transcendental. Making use of this fact and of Hermite's method of proof, Lindemann, in 1882 proved that π was transcendental, settling forever the famous problem.

The book is well written and is well worth being read by every teacher of plane geometry.

FRANK M. MORGAN.

NOTES.

The twenty eighth summer meeting of the American Mathematical Society will be held at Wellesley College, Wellesley, Massachusetts, on Wednesday, Thursday, and Friday, September 7–9. It will be preceded by the summer meeting of the Mathematical Association of America. A joint session of the two organizations has been arranged for the afternoon of Wednesday, September 7, with two speakers on phases of the Theory of Relativity. The usual joint dinner will be on Wednesday evening.

The regular sessions will be held in Founders' Hall. Tower Court will be open for the accommodation of members and friends, a separate wing being assigned to ladies and married couples. The Hall in this building will be available for social purposes during the meetings. General social functions and an excursion will be arranged.

Wellesley is fifteen miles from Boston, on the main line of the Boston and Albany Railroad. There is frequent train service from Boston as well as from Worcester and the West.

Titles and abstracts of papers intended for presentation at the meeting should be in the hands of the secretary by August 8, in order to appear on the printed program. Papers received after that date can be presented only as time permits. This announcement replaces the preliminary notice issued in the past. The attention of members is invited to the fact that no other preliminary notice will be issued.

The Division of Physical Sciences of the National Research Council was held in Washington on Wednesday, April 20. The following officers were elected for the ensuing year: chairman, Professor H. G. Gale; vice-chairman, professor E. P. Lewis; executive committee, Professors R. A. Millikan, Augustus Trowbridge, O. Veblen, John Zelleny. As members-at-large, Professors E. F. Nichols, S. W. Stratton, and G. W. Stewart were elected. The appointment of Professor E. B. van Vleck to succeed Professor H. S. White as one representative of the American Mathematical Society was announced. Professor L. E. Dickson reported for the American Branch of the International Mathematical Union that an invitation had

been extended by the American Branch at the Strasbourg meeting to hold the next International Congress of Mathematics in America, and that this invitation had been accepted. It is therefore presumed that such a Congress will be held in America in the year 1924 or as soon thereafter as is feasible. A similar report was made by Professor Dickson to the Council of the American Mathematical Society at its meeting on Friday evening, April 22, in New York.

In January, 1921, the National Association of German Mathematical Societies and Clubs was founded at Göttingen, the charter members including the German Mathematical Society, the Society for the Advancement of Mathematical and Scientific Education, the Mathematico-Scientific Society of Württemburg, and the mathematical societies of Berlin, Vienna, Hamburg, Göttingen, and Aachen. It is desired to unite in this association all mathematical societies of German speaking countries, whether their aim is primarily educational or scientific.

The newly founded Vereinigung der Freunde und Förderer des Positivistischen Idealismus announces in its organ, the ANNALEN DER PHILOSOPHIE, a prize of 5000 Marks for a paper on the following subject: *The relation of Einstein's relativity theory to modern philosophy, with special reference to the philosophy of "As-If."* Further information may be obtained from the editor of the ANNALEN DER PHILOSOPHIE, Dr. Raymond Schmidt, Fichtestrasse 13, Leipzig.

The American Association for the Advancement of Science has granted the sum of one hundred fifty dollars to Professor S. Lefschetz, of the University of Kansas, in support of his work in algebraic geometry.

Cambridge University has awarded Smith's prizes to L. A. Pars, of Jesus College, for an essay on *The general theory of relativity*, and to W. M. H. Greaves, for an essay on *Periodic orbits in the problem of three bodies.*

The Albert medal of the Royal Society of Arts has been conferred on Professor A. A. Michelson, of the University of Chicago, for his discovery of a natural constant which has provided a basis for a standard of length.

The University of Dublin has conferred honorary doctorates of science on Professor Emile Borel, of the Sorbonne, and Professor A. A. Michelson, of the University of Chicago.

Professor E. W. Brown, of Yale University, has been elected correspondent of the Paris Academy of Sciences in the section of astronomy, as successor to the late Professor E. C. Pickering, and Sir Ernest Rutherford has been elected correspondent in the section of physics, as successor to Professor A. A. Michelson, elected foreign associate.

At the University of Bonn, Associate Professor H. Beck has been promoted to a full professorship of mathematics. Professor H. Hahn has resigned, to accept a professorship of mathematics at the University of Vienna.

Professor L. Bieberbach, of the University of Frankfurt a. M., has been appointed successor to Professor C. Carathéodory at the University of Berlin.

Dr. H. Dingler and Dr. A. Rosenthal have been promoted to associate professorships at the University of Munich.

Professor E. Hellinger has been promoted to a full professorship of mathematics at the University of Frankfurt a. M.

Dr. L. Lichtenstein has been appointed honorary professor of mathematics at the Berlin Technical School.

At the Darmstadt Technical School, Dr. E. Naetsch has been appointed full professor of analytic geometry. Professor L. Henneberg has retired from active teaching.

Dr. J. Nielsen, of the University of Hamburg, has been appointed to a professorship at the Breslau Technical School.

Professor L. Prandtl, of the University of Göttingen, has been appointed professor of technical mechanics at the Munich Technical School.

Professor J. Thomae, of Jena, celebrated his eightieth birthday on December 11, 1920.

Dr. V. Geilen has been appointed lecturer in applied mathematics at the University of Münster.

Dr. W. Lorey has been appointed lecturer on the mathematics of insurance at the University of Leipzig.

Dr. W. Sternberg has been appointed privat docent in mathematics at the University of Heidelberg.

Dr. G. Pólya has been promoted to a professorship at the Zurich Polytechnikum.

Professor René Granier, of the University of Poitiers, has

at his own request been transferred from the chair of theoretical and applied mechanics to that of the differential and integral calculus.

Professor A. S. Eddington has been elected president of the Royal Astronomical Society.

The council of St. John's College, Cambridge, has appointed Dr. T. J. I'A. Bromwich prælector in mathematical science.

Associate Professor W. A. Manning has been promoted to a full professorship of mathematics at Stanford University.

Professor Frank Morley of Johns Hopkins University is now in England on leave of absence for the second semester of the current year.

Professor Albert Einstein will deliver a course of five lectures on relativity and gravitation theory at Princeton University on May 9–13. He also delivered lectures at Columbia University on April 16, and at the College of the City of New York on April 18–21.

Professor A. von Oettingen died at the age of eighty-four years.

The death is reported of Professor M. Petzold, of the Hanover Technical School.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Amodeo (F.). Le ricerche di un matematico napoletano del settecento su alcune teoremi di Archimede e sulle loro estensioni. Napoli, R. Accademia delle Scienze fisiche e matematiche, 1920.

Behrens (G.). Die Prinzipien der mathematischen Logik bei Schröder, Russell und König. (Diss., Kiel.) Hamburg, Druck von Berngrüber und Henning, 1918. 8vo. 65 pp.

Fawdry (R. C.). Coordinate geometry (plane and solid) for beginners. London, Bell, 1921. 8 + 215 pp. 5s.

Glause (H.). Geometrischer Beweis der Ergänzungssätze zum bikubischen Reziprozitätsgesetz. (Diss., Kiel.) Altona, Druck von Hammerich und Lesser, 1918. 8vo. 61 pp.

Hagström (K. G.). Der Begriff der statistischen Funktion. (Diss.) Uppsala, Almqvist och Wiksell, 1919. 8vo. 52 pp.

HEIBERG (J. L.). Naturwissenschaften, Mathematik und Medizin im klassischen Altertum. 2te Auflage. (Aus Natur und Geisteswelt, Nr. 370.) Leipzig, Teubner, 1920. 104 pp.

HOBSON (E. W.). The theory of functions of a real variable and the theory of Fourier's series. 2d edition. Volume 1. Cambridge, University Press, 1921. 16 + 671 pp. 45s.

KLEIN (L.). Streifzüge in das Gebiet der Mathematik und Geometrie. Korneuberg, Fellner und Zausner, 1920.

PALMQUIST (R.). Quelques études sur la convergence des déterminants infinis et les systèmes d'une infinité d'équations linéaires à une infini d'inconnus. (Diss.) Uppsala, Almqvist och Wiksell, 1915. 8vo. 52 pp.

PERL (A.). Ueber die singulären Punkte der algebraischen Flächen dritter Ordnung. Leipzig, Teubner, 1915. 8vo. 46 pp.

PRÜSS (A.). Ueber vollständig zerfallende algebraische Raumkurven und Flächen. (Diss., Kiel.) Kiel, Schmidt und Klaunig, 1918. 8vo. 55 pp.

REIMERS (M. H.). Zur Theorie der algebraischen Raumkurven und ihrer Tangentenflächen. (Diss., Kiel.) Leipzig, Teubner, 1915. 8vo. 55 pp.

SIMPSON (T. McN.). Relations between the metric and projective theories of space curves. (Diss., Chicago.) Private edition, distributed by the University of Chicago Libraries, 1920. 4to. 26 pp.

DE VRIES (H.). Leerboek der differentiaal- en integraalrekening en van de theorie der differentiaalvergelijkingen. 2 deele. Groningen, Noordhoff, 1919–1920. 750 + 460 pp. f. 19.20 + 16.50

II. ELEMENTARY MATHEMATICS.

BARDEY (E.). Aufgabensammlung, methodisch geordnet. Neue Ausgabe nach F. Pietzker und O. Presler bearbeitet von A. Mohrmann. 10te Auflage. Leipzig, Teubner, 1920. 450 pp.

BOJKO (J.). Lehrbuch der Rechenvorteile. Schnellrechnen und Rechenkunst. (Aus Natur und Geisteswelt, Nr. 739.) Leipzig, Teubner, 1920. 115 pp.

DIECK (W.). Mathematisches Lesebuch. 1ter und 2ter Band. Sterkrade, Osterkamp, 1920.

DURELL (C. V.). A concise geometry. London, Bell, 1921. 8 + 319 pp. 5s.

GOLLER (A.). See Lengauer (J.).

GONGGRIJP (B.). Zaktafels in vier decimalen. Groningen, Noordhoff, 1919. 52 pp.

VAN DE GRIEND (J.). Trigonometrische vraagstukken met beknopte theorie. Antwoorden op de Trigonometrische vraagstukken met beknopte theorie. Groningen, Noordhoff, 1920. 8vo. 132 + 24 pp. f. 1.90 + 1.00

LENGAUER (J.). Die Grundlehren der ebenen Geometrie. 20te Auflage, von A. Goller. Kempten, Kösel, 1920. 160 pp.

MOHRMANN (A.). See BARDEY (E.).

PIETZKER (F.). See BARDEY (E.).

PRESLER (O.). See BARDEY (E.).

VAN THIJN (A.). Leerboek der vlakke meetkunde met opgaven. Eerste deel, vijfte druk. Groningen, J. B. Wolters, 1920. 8vo. 119 pp. f. 2.40

WEBER (M.). Les mathématiques de l'élève ingénieur. I: Algèbre et analyse. 1re partie: Notions fondamentales. Trigonométrie. Algèbre. Paris, Dunod, 1920. 8vo. 14 + 495 pp. Fr. 45.00

III. APPLIED MATHEMATICS.

ABRAHAM (M.). Theorie der Elektrizität. 2ter Band: Elektromagnetische Theorie der Strahlung. 4te Auflage. Leipzig, Teubner, 1920.

D'ADHÉMAR (R.). Résistance des matériaux. Paris, Gauthier-Villars, 1921. 8vo. 9 + 185 pp. Fr. 20.00

BOLZMANN (L.). See BUCHHOLZ (H.).

BORN (M.). Die Relativitätstheorie Einsteins und ihre physikalischen Grundlagen. Gemeinverständlich dargestellt. (Naturwissenschaftlichen Monographien und Lehrbücher, Band 3.) Berlin, Springer, 1920.

BOSE (S. N.). See EINSTEIN (A.).

BOUASSE (H.). Statique. Machines simples, bascules et balances, frottement, graissage, statique graphique. Paris, Delagrave, 1920. 8vo. 24 + 515 pp.

BRYANT (W. W.). Kepler. (Pioneers of Progress: Men of Science Series.) London, Society for the Promotion of Christian Knowledge, and New York, Macmillan, 1921. 12mo. 62 pp. 2s.

BUCHHOLZ (H.). Ludwig Boltzmanns Vorlesungen über die Prinzipe der Mechanik. 3ter Teil: Elastizitätstheorie und Hydromechanik. Leipzig, Barth, 1920. 820 pp.

CASE (J.). Theory of direct current dynamos and motors. Cambridge, Heffer, 1921. 13 + 196 pp. 15s.

CLAUZEL (G.). Effets gyroscopiques et méthodes vectorielles. Paris, Dunod, 1920. 8vo. 147 pp. Fr. 14.00

DISPER (P.). Ueber die Massenverteilung und Verschiebung der Druck- und Zugkräfte in einem Kometen. Montabaux, Kalb, 1919. 42 pp.

EGERER (A.). Kartenkunde. I: Einführung in das Kartenverständnis. (Aus Natur und Geisteswelt, Nr. 610.) Leipzig, Teubner, 1920. 146 pp.

EINSTEIN (A.) and MINKOWSKI (H.). The principle of relativity. Original papers by A. Einstein and H. Minkowski, translated into English by M. N. Saba and S. N. Bose. Calcutta, University Press, 1921. 23 + 186 pp.

ENGLUND (E.). Sur les méthodes d'intégration de Lie et les problèmes de mécanique céleste. (Diss.) Uppsala, Almqvist och Wiksell, 1916. 8vo. 50 pp.

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The two hundred sixteenth regular meeting of the Society was held at Columbia University on Saturday, April 23, 1921, extending through the usual morning and afternoon sessions. The attendance included the following sixty-seven members:

Professor R. C. Archibald, Professor R. A. Arms, Dr. Charlotte C. Barnum, Professor A. A. Bennett, Professor E. G. Bill, Professor G. D. Birkhoff, Professor H. F. Blichfeldt, Dr. R. F. Borden, Professor R. W. Burgess, Professor Abraham Cohen, Dr. G. M. Conwell, Professor Louise D. Cummings, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Jesse Douglas, Professor L. P. Eisenhart, Professor W. B. Fite, Mr. R. M. Foster, Mr. Philip Franklin, Mr. B. P. Gill, Dr. T. H. Gronwall, Dr. C. C. Grove, Dr. C. M. Hebbert, Professor E. R. Hedrick, Lieutenant R. S. Hoar, Professor L. S. Hulburt, Professor W. A. Hurwitz, Mr. S. A. Joffe, Professor Edward Kasner, Professor O. D. Kellogg, Dr. E. A. T. Kircher, Professor J. R. Kline, Mr. Harry Langman, Professor Florence P. Lewis, Mr. L. L. Locke, Mr. John McDonnell, Professor H. F. MacNeish, Professor L. C. Mathewson, Professor H. H. Mitchell, Professor G. W. Mullins, Dr. Almar Naess, Professor E. J. Oglesby, Professor W. F. Osgood, Professor F. W. Owens, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor L. W. Reid, Dr. C. N. Reynolds, Mr. L. H. Rice, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. J. E. Rowe, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor L. L. Silverman, Professor P. F. Smith, Dr. J. M. Stetson, Miss Louise E. C. Stuerm, Professor K. D. Swartzel, Mr. H. S. Vandiver, Professor J. H. M. Wedderburn, Professor Mary E. Wells, Dr. Norbert Wiener, Miss Ella C. Williams, Professor H. S. White, Professor J. K. Whittemore, Professor Ruth G. Wood.

Professors L. E. Dickson and W. F. Osgood presided at the morning session, and Professor P. F. Smith, relieved by Professors H. S. White and Abraham Cohen, in the afternoon. The afternoon session opened with a one-hour lecture given by Professor W. A. Hurwitz at the invitation of the programme committee, on *Topics in the theory of divergent series.*

The Council announced the election of the following persons to membership in the Society:

Mr. Lewis Albert Anderson, Central Life Assurance Society, Des Moines;
Dr. Eugene Manasseh Berry, Purdue University;
Mr. Raymond Van Arsdale Carpenter, Metropolitan Life Insurance Company, New York;
Mr. James Douglas Craig, Metropolitan Life Insurance Company, New York;
Mr. Bernard Francis Dostal, Oberlin College;
Mr. George Graham, Central States Life Insurance Company, St. Louis;
Dr. Jacob Millison Kinney, Hyde Park High School, Chicago;
Mr. John Ruse Larus, Jr., Phoenix Mutual Life Insurance Company, Hartford;
Professor Daniel Acker Lehman, Goshen College;
Mr. Joseph Brotherton Maclean, Mutual Life Insurance Company, New York;
Mr. Franklin Bush Mead, Lincoln National Life Insurance Company, Fort Wayne;
Professor James Newton Michie, University of Texas;
Mr. Henry Moir, Home Life Insurance Company, New York;
Mr. William Oscar Morris, North American Life Insurance Company, Chicago;
Mr. Oliver Winfred Perrin, Penn Mutual Life Insurance Company, Philadelphia;
Professor Harris Rice, Worcester Polytechnic Institute;
Miss Jessie M. Short, Reed College;
President Wen Shion Tsu, Nanyang Railway and Mining College;
Professor Buz M. Walker, Mississippi Agricultural and Mechanical College;
Mr. Robert Montague Webb, Kansas City Life Insurance Company;
Mr. Archibald Ashley Welch, Phoenix Mutual Life Insurance Company, Hartford;
Professor Frank Edwin Wood, Michigan Agricultural College;
Mr. Joseph Hooker Woodward, Equitable Life Assurance Society, New York;
Mr. William Young, New York Life Insurance Company.

Eleven applications for membership were received.

It was voted by the Council to accept the invitations received at the preceding meeting to hold the next annual meeting of the Society at Toronto during the Christmas holidays in connection with the meeting of the American Association for the Advancement of Science, and to hold a meeting of the Society with the Chicago Section at Chicago in 1922. The time of the Chicago meeting was not determined.

Professor G. D. Birkhoff was reelected a member of the Editorial Committee of the TRANSACTIONS for a term of three years beginning October 1, 1921.

Professor P. F. Smith presented to the Council the recommendations of the committee authorized at the last meeting to consider policies for conserving the interests of the Society in foreign countries for the year 1921. These recommendations

were adopted and the committee discharged with the thanks of the Council. It was voted to request the President to appoint a committee to consider and report to the Council a policy for 1922 with regard to these and related matters. A committee was also authorized to prepare nominations for officers and other members of the Council to be elected at the annual meeting in December.

It was decided that on the request of the contributor twenty-five reprints (with covers) of articles in the BULLETIN be given without charge, and that a statement be made to authors concerning the cost of reprints with a request that they forego this privilege of free copies unless they can make good use of them.

The usual luncheon and dinner were held at the Faculty Club.

The titles and abstracts of the papers read at this meeting follow below. Professor Huntington's paper was read by Professor Kellogg, and Professor Schwatt's papers by Professor Kline; the papers of Professors Poor, Lipka, Jackson, and McMahon, Dr. Walsh, Professor Fischer, Dr. Post and Professor Coble were read by title.

1. Professor W. F. Osgood: *On the gyroscope.*

When a gyroscope moves under the action of any forces, its axis describes a cone at a definite rate. The discovery of a form of the intrinsic relation between these three factors is the chief result of this paper. Since one point, O, in the axis is fixed, the applied forces (the constraint at O being omitted) can be replaced (a) by a single force, $\mathfrak{F}$, acting at a point ζ of the axis at unit distance from O and at right angles to $O\zeta$; (b) by a couple, $\mathfrak{N}$, whose plane is perpendicular to the axis. The contribution of the cone, $\mathfrak{K}$, lies in its *bending*, κ. By this is meant the rate at which the tangent plane at O turns when ζ describes its path, $\mathfrak{C}$, on the unit sphere with unit velocity. Finally, let v denote the velocity with which ζ describes its path $\mathfrak{C}$ in the case of the actual motion, and let r denote the angular velocity of the gyroscope about its axis. Then

$$Av\frac{dv}{ds} = T, \qquad A\kappa v^2 + Crv = Q, \qquad C\frac{dr}{dt} = N,$$

where s denotes the arc of $\mathfrak{C}$, T and Q the components of $\mathfrak{F}$ along the tangent to $\mathfrak{C}$ and the normal perpendicular to $O\zeta$

respectively, and C and A are the moments of inertia about the axis $O\zeta$ of the gyroscope and an axis through O perpendicular to $O\zeta$. The foregoing relations are *intrinsic*, i.e. independent of any particular choice of coordinates for the gyroscope. They apply directly to the motion of the axis. In the case most important in practice, $N = 0$, and the angular velocity, r, reduces to a constant, ν. There remain, then, merely the first two equations, in which r is replaced by ν.

2. Professor H. S. White: *Seven points in space and the eighth associated point.*

The construction given by O. Hesse in CRELLE'S JOURNAL, vol. 20, is interwoven with the relations of the eight points to three quadric surfaces on which they lie, and to one auxiliary quadric. Using the same construction by points and lines, this paper avoids all reference to quadrics. By algebraic formula in invariant form it proves the uniqueness of the derived eighth point and the symmetry of the completed set of eight, and it gives explicitly the equation of the derived point, covariant, of degree seven, in the given points.

3. Professor L. E. Dickson: *Most general composition of polynomials.*

This paper treats the problem to find three polynomials $f(x) \equiv f(x_1, \cdots, x_n)$, ϕ, F such that $f(x)\phi(\xi) = F(X)$, identically in $x_1, \cdots, x_n, \xi_1, \cdots, \xi_n$, when $X_1, \cdots, X_n$ are bilinear functions of these $2n$ independent variables. The author proves that this problem reduces to the case of a single polynomial f for which $f(x)f(\xi) = f(X)$. This case was treated by him in the COMPTES RENDUS DU CONGRÈS INTERNATIONAL DES MATHÉMATICIENS, Strasbourg, 1920, pp. 131–146. The new paper will appear in the COMPTES RENDUS.

4. Professor L. E. Dickson: *Number of real roots by Descartes's rule of signs.*

In this paper, the author gives a complete, elementary proof of Descartes's rule of signs for the number of real roots of an equation $f(x) = 0$ with real coefficients. The proof follows from the following fact, which is proved by induction. The number of variations of signs in the set $a_0, a_1, \cdots, a_n$ exceeds by a positive even integer the number of variations of signs in $b_0, \cdots, b_{n-1}$, if b_0 is any chosen number of the same

sign as a_0, while the remaining b's are found as in the process of synthetic division by means of a positive multiplier.

5. Professor L. P. Eisenhart: *The Einstein solar field.*

This paper appears in full in the present number of this BULLETIN.

6. Professor J. K. Whittemore: *A special kind of ruled surface.*

In this paper are determined all ruled surfaces having the property that every pair of curved asymptotic lines cuts a constant length from a variable ruling. It is shown that these surfaces have an analogy with the right helicoid; in particular that all rulings of such a surface are parallel to a plane and that the parameter of distribution of such a surface is constant. It is proved that a surface of this type is completely determined by the choice of a plane curve, a straight line, and the constant parameter of distribution. Certain questions regarding these surfaces are considered.

7. Professor V. C. Poor: *On the theorems of Green and Gauss.*

This paper contains a definition for the divergence of a vector in terms of the point differential, namely:

$$\operatorname{div} u\, dP \times \delta P \wedge \vartheta P = du \times \delta P \wedge \vartheta P + \delta u \times \vartheta P \wedge dP + \vartheta u \times dP \wedge \delta P,$$

with the proof for the uniqueness and the existence of the idea so defined. The statement and proof of the following very general form of Green's theorem is given:

$$\int I_1 \left(K\alpha \frac{d \operatorname{grad} \beta}{dP} - K\beta \frac{d \operatorname{grad} \alpha}{dP} \right) d\tau = \int (K\alpha \operatorname{grad} \beta - K\beta \operatorname{grad} \alpha) \times u d\sigma.$$

Another theorem states that

$$\int \left(\frac{d\alpha}{dP} u \right) x d\tau = - \int u \times u \cdot \alpha x d\sigma - \int \alpha \frac{dx}{dP} u d\tau.$$

8. Dr. J. E. Rowe: *Pressure distribution around a breech-block.*

The breech-block of a certain gun is circular in form and is held in place by a thread on its circumference. When the

gun is fired the gas presses against a circular portion of this breech-block, whose area is equal to the cross-sectional area of the powder chamber of the gun, and which is eccentrically situated with respect to the breech-block. The purpose of the paper is to derive the intensity of the pressure on any part of the thread from a given powder pressure.

9. Professor E. V. Huntington: *The mathematical theory of proportional representation. Third paper.*

The author's method of apportionment of representatives, which was presented at the December and February meetings, and which is now known as the "method of equal proportions," solves the problem of apportionment by a direct and obvious comparison between the several states, without the use of the idea of total error. A satisfactory expression for total or average error may, however, be stated, as follows: $E = \sqrt{S/N}$, where $S = \Sigma[(a - \alpha)^2/a]$. Here a is the actual and α the theoretical number of representatives assigned to a typical state A, and N is the total number of representatives. This quantity E (or S) is the quantity which is minimized by the method of equal proportions. The quantity Q, proposed at the February meeting by Professor F. W. Owens, and minimized by the Willcox method of major fractions, may be written as $Q = \Sigma[(a - \alpha)^2/\alpha]$. These quantities S and Q and other similar expressions are compared, and reasons given for preferring S.

10. Professor F. W. Owens: *On the apportionment of representatives. Second paper.*

This paper is supplementary to the author's paper on the same subject read at the February meeting of the Society and contains further developments and illustrations of the results of the different methods.

11. Professor Joseph Lipka: *On the geometry of motion in a curved space of n dimensions.*

In this paper are derived a number of geometric properties of certain systems of ∞^{2n-1} curves, termed q systems, in a curved space of n dimensions. These systems include dynamical trajectories, brachistochrones, catenaries, and velocity curves, under conservative forces or under arbitrary positional fields of force as special cases. The properties are

expressed in terms of osculating and hyperosculating geodesic surfaces and hyperosculating geodesic circles and the loci of their centers of curvature.

12. Professor Dunham Jackson: *Note on an irregular expansion problem.*

This paper will appear in full in the January number of this BULLETIN.

13. Professor James McMahon: *Hyperspherical goniometry, with applications to the theory of correlation for n variables.*

The chief suggestion for this paper is contained in an article by Karl Pearson in BIOMETRIKA (vol. 11, p. 237). In Part I, the formulas of spherical trigonometry are generalized for the hypersphere in n dimensions, the underlying hypergeometry being first sketched in a form adapted to the purpose in hand. In Part II, a far-reaching connection between the geometry of the hypersphere and the theory of correlation for n variables is established by linearly transforming the 'ellipsoids' of equal frequency (on an n-dimensional correlation chart) into hyperspherical surfaces. The generalized formulas of hyperspherical goniometry are then applied to furnish easy proofs of various new and old theorems in multiple and partial correlation (including the relations referred to above), and to point the way to further developments.

14. Dr. J. L. Walsh: *On the location of the roots of polynomials.*

This paper contains a proof of the following theorem: If the coefficients of a polynomial $f(z)$ are linear in each of the parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$ and are symmetric (but not necessarily homogeneous) in these parameters, and if the points α lie in a circular region C, then for any value of z we may replace the α's by k parameters *which coincide in* C and this without changing the value of $f(z)$.

This theorem has various applications; it can be easily proved that if all the k roots of a polynomial $\varphi(z)$ lie on or within a circle C whose center is α and radius r, then all the roots of

$$A_0\varphi(z) + A_1\varphi'(z) + A_2\varphi''(z) + \cdots + A_k\varphi^{(k)} = 0$$

(when the A's are arbitrary constants) lie on or within the circles of radius r whose centers are the roots of

$$A_0(z-\alpha)^k + kA_1(z-\alpha)^{k-1} + k(k-1)A_2(z-\alpha)^{k-2} + \cdots + k(k-1)\cdots 1\cdot A_k = 0.$$

15. Professor C. A. Fischer: *The kernel of the Stieltjes integral corresponding to a completely continuous transformation.*

In the 1918 ACTA MATHEMATICA, F. Riesz has proved that a linear transformation which is completely continuous, $A[f]$, can be decomposed into the sum of two orthogonal transformations, $A_1[f]$ and $A_2[f]$, such that the transformation $B_1[f] = f(x) - A_1[f]$ has an inverse, and the equations $B_2^n[\varphi] = 0$ and $B^n[\varphi] = 0$ have the same solutions. He has also proved in an earlier paper that every linear transformation can be put into the form

$$A[f] = \int_a^b f(y)d_yK(x, y).$$

In this BULLETIN, October, 1920, Professor Fischer gave the conditions which $K(x, y)$ must satisfy in order that $A[f]$ be completely continuous. In the present paper he has proved that the function $K_2(x, y)$, corresponding to Riesz' $A_2[f]$, can be put into the form

$$K_2(x, y) = \sum_{i=1}^{n} \varphi_i(x)\psi_i(y),$$

where the functions $\varphi_i(x)$ are solutions of the equation $B^\nu[\varphi] = 0$, ν being a positive integer such that every solution of $B^{\nu+1}[\varphi] = 0$ is also a solution of $B^\nu[\varphi] = 0$. The applications of this theorem to Stieltjes' integral equations of both the first and second kinds are then discussed.

16. Dr. E. L. Post: *On a simple class of deductive systems.*

In the present paper the author considers a general class of deductive systems involving primitive functions of but one argument, and solves for all such systems the following problem: to find a method for determining in a finite number of steps whether a given enunciation of the system can or cannot be asserted by means of the postulates of the system. The solution is obtained by directly analyzing the way in which assertions are generated from the primitive assertions by the rules of deduction of the system, and gives the beginning of a

distinct alternative to the truth-table method which was introduced in its simplest form in a previous paper.

17. Professor W. A. Hurwitz: *Topics in the theory of divergent series.*

This paper will appear in full in the January number of this BULLETIN.

18. Dr. Norbert Wiener and Professor F. L. Hitchcock: *A new vector method in integral equations.*

Starting with the Fredholm equation

$$u(x) - \lambda \int_a^b u(y)K(x, y)dy = v(x)$$

the authors say that $f(x)$ is *conjoint* to $g(x)$ if

$$f|g = \int_a^b f(x)g(x)dx - \lambda \int_a^b \int_a^b f(x)g(y)K(x, y)dxdy = 0.$$

They then develop methods of rendering a closed set of functions conjoint by pairs, and they show that if $\{\varphi_n\}$ is a closed conjoint set of functions and

$$\sum_n \varphi_n(x) \frac{\int_a^b v(x)\varphi_n(x)dx}{\varphi_n | \varphi_n}$$

converges uniformly, it represents a solution of the Fredholm equation. Further extensions of this method give series for u that converge in the mean, and enable the determination of the characteristic numbers.

The homogeneous equation

$$u(x) - \lambda \int_a^b u(y)K(x, y)dy = 0$$

is solved by the determination of a function orthogonal to every

$$f(x) - \lambda \int_a^b f(y)K(y, x)dy.$$

A similar method is developed for the solution of the non-homogeneous equation.

All the methods of the present paper are immediate generalizations of vector methods.

19. Dr. Jesse Douglas: *On a certain type of system of* ∞^2 *curves.*

The ∞^2 straight lines of the plane form a family in which the sum of the angles of every triangle is equal to two right angles. The most general families of ∞^2 curves having the same property are the loxodromes, that is those formed of the totality of isogonal trajectories of a given simply infinite system of curves. These are characterized by differential equations of the form

$$y'' = (A + By')(1 + y'^2),$$

where A and B are any functions of x and y such that

$$A_y - B_x = 0.$$

A natural generalization is to inquire as to the systems of ∞^2 plane curves which are such that in every triangle formed of three curves of the system, the sum of the angles differs from π by an amount proportional to the area. The author bases the analysis on the equations of variation, showing that it is sufficient to restrict consideration to triangles of which two sides are formed of infinitesimally adjacent curves, while the third is finitely divergent from these two. The result is that the differential equation of the family must be of the form

$$y'' = (A + By')(1 + y'^2),$$

with $A_y - B_x = k$, where k is the constant of proportionality. A synthetic construction for curve systems of the type so defined is obtained by starting with any system of ∞^1 curves, and drawing trajectories to cut them, not under constant angle, but so that the angle increases by k times the element of area ydx under the trajectory.

The above considerations are easily extended to systems of curves on a general surface.

20. Dr. Jesse Douglas: *Concerning Laguerre's inversion.*

In a previous paper (this BULLETIN, July, 1920) the author made use of a certain representation of the (directed) lines of the plane on the cylinder $x^2 + y^2 = 1$; namely, the line of Hessian coordinates (ω, p) corresponding to the point $(x, y, z) = (\cos \omega, \sin \omega, p)$. The aspect of this representation fundamental for the present paper is that the (oriented)

circles of the plane correspond to the plane sections of the cylinder. By a projective transformation of space the cylinder may be replaced by a general quadric cone. It results that the line transformations in the plane that convert circles into circles form a group, of seven parameters, of which the geometry is identical with projective geometry on a quadric cone.

The last statement is dual to the fact that the point transformations preserving cocircularity form a six-parameter group of which the geometry is identical with projective geometry on a sphere (or any ellipsoid), the identity between the two geometries being based on stereographic projection, which represents the circles of the plane as plane sections of the sphere. The representation described at the beginning thus appears as the dual of stereographic projection.

In the above G_6 of point transformations are included the ordinary inversions, which may be distinguished as corresponding to the perspectives of the sphere. On inquiring what corresponds in the G_1 of line transformations to the perspectives of the cylinder, one comes precisely upon a certain geometric construction described by Laguerre in a paper of 1882 (*Oeuvres*, vol. 2, p. 611). Laguerre studies the properties of his transformation from various points of view, bringing out especially the duality to ordinary inversion, but the aspects above developed appear to be new.

21. Professor J. R. Kline: *Closed connected point sets which are disconnected by the omission of a finite number of points.*

Suppose M is a closed connected point set such that if P is any point of M,

$$M - P = M_1 + M_2,$$

where M_1 and M_2 are two mutually exclusive sets neither of which contains a limit point of the other one. The author shows that M is a continuous curve in the sense of R. L. Moore, i.e. it is closed connected, and connected im kleinen. Suppose G is a closed connected set such that (1) if P is any point in G, $G - P$ is connected, (2) if P_1 and P_2 are any two distinct points of G, we have

$$G - P_1 - P_2 = G_1 + G_2,$$

where G_1 and G_2 are two mutually exclusive sets neither of which contains a limit point of the other one. It is shown that G is a simple closed curve.

22. Professor I. J. Schwatt: *The sum of a series as the solution of a differential equation.*

Boole (*A Treatise on Differential Equations,* 3d edition, pp. 441–450) considers the summation of series that are the solutions of differential equations of the special form

$$\sum_{k=0}^{n} f_k(D)e^{k\theta}u = 0.$$

In this paper a general method for the summation of series as the solutions of linear differential equations is developed and methods are given for solving the equations.

23. Professor I. J. Schwatt: *Method for the summation of a general case of a deranged series.*

Dirichlet, Riemann, Pringsheim and Scheibner have shown that if the terms of a convergent series are deranged, the sum of the resulting series is, in general, different from the sum of the given series. The author has treated two particular cases of deranged series (ARCHIV DER MATHEMATIK UND PHYSIK, vol. 24 (1915), p. 139 and GIORNALE DI MATHEMATICHE, vol. 54 (1916)) and considers in this paper the sum of a general case of a deranged series.

24. Professor I. J. Schwatt: *Higher derivatives of functions of functions.*

Several methods for finding the higher derivatives of functions of functions have been given but they are not in a form convenient for the purposes of application. To this seems to be due the fact that even the leading treatises on the calculus give the higher derivatives of only the simplest functions of functions, in most cases derived by special devices or by induction. Also in the expansion of functions these authors and others find as a rule by actual differentiation the first few derivatives and correspondingly obtain the first few terms of the expansion. In the course of this paper specific references are given. The author has developed methods which enable him to find the general higher derivative of functions of functions and their complete expansions.

25. Professor A. B. Coble: *A covariant of three circles.*

This paper appears in full in the present number of this BULLETIN.

R. G. RICHARDSON,
Secretary.

THE EASTER MEETING OF THE SOCIETY AT CHICAGO.

THE forty-seventh regular meeting of the Chicago Section, constituting the sixteenth regular Western meeting of the Society, was held at the University of Chicago on Friday and Saturday, March 25 and 26, 1921, the first session opening at 10 A.M. on Friday in Room 32, Ryerson Physical Laboratory.

Over sixty persons were present at this meeting, among whom were the following fifty-three members of the Society: Miss Mary C. Ball, Professor G. A. Bliss, Professor Henry Blumberg, Professor R. L. Borger, Dr. H. R. Brahana, Professor C. C. Camp, Professor R. D. Carmichael, Professor E. W. Chittenden, Dr. H. B. Curtis, Professor W. W. Denton, Professor L. E. Dickson, Professor Arnold Dresden, Professor John Eiesland, Professor Arnold Emch, Professor W. B. Ford, Dr. Gladys Gibbens, Dr. Josephine B. Glasgow, Professor W. L. Hart, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor T. F. Holgate, Professor Dunham Jackson, Miss Claribel Kendall, Professor A. M. Kenyon, Professor W. C. Krathwohl, Professor Kurt Laves, Mrs. Mayme I. Logsdon, Professor A. C. Lunn, Professor W. D. MacMillan, Professor T. E. Mason, Professor E. D. Meacham, Professor W. L. Miser, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Dr. C. A. Nelson, Professor C. I. Palmer, Professor Anna H. Palmié, Professor H. L. Rietz, Mr. Irwin Roman, Professor J. B. Shaw, Professor H. E. Slaught, Dr. L. L. Steimley, Mr. E. L. Thompson, Professor E. J. Townsend, Professor E. B. Van Vleck, Professor G. E. Wahlin, Professor E. J. Wilczynski, Professor C. E. Wilder, Professor R. E. Wilson, Professor C. H. Yeaton, Professor A. E. Young, Professor J. W. A. Young.

Forty-seven persons attended the dinner held at the Quadrangle Club on Friday evening.

At the session of Friday afternoon, presided over by President Bliss, Professor Dunham Jackson read an expository paper on *The general theory of approximation by polynomials and trigonometric sums*. This paper appears in the present number of this BULLETIN. At the close of the discussion, which followed the presentation of this paper, Professor

Wilczynski moved and the Section adopted unanimously a resolution expressing to Professor Jackson their appreciation of the admirable way in which he had presented his subject. In view of the great value of the expository work done by Professor de la Vallée Poussin in the field discussed by Professor Jackson, an address expressing to Professor de la Vallée Poussin their appreciation of his work was signed by those present at the dinner on Friday evening.

The other sessions were presided over by Professor Carmichael, Chairman of the Section, relieved for part of Saturday forenoon by Vice-President Jackson.

The papers presented at the meeting are stated below. Professor Schwatt's papers were communicated to the Society by Professor Dunham Jackson. The papers of Professors Miller and Lane were read by title.

1. Professor I. J. Schwatt: *On the expansion of powers of trigonometric functions.*

No satisfactory expressions for the expansions of powers of trigonometric functions seem to have been given. Only two articles on the subject have been found. One, by Dr. Ely in the AMERICAN JOURNAL OF MATHEMATICS, vol. 5 (1882), p. 359, treats only of the expansion of $\sec^p x$ for odd powers of p and obtains by induction a result which involves Euler's numbers. The other, by Shovelton, in the QUARTERLY JOURNAL OF MATHEMATICS, vol. 46 (1915), pp. 220–247, uses the theory of finite differences.

The author has obtained expansions which are believed to be simple and new.

2. Professor I. J. Schwatt: *On the summation of a trigonometric power series.*

To obtain the sum

$$S \equiv \sum_{n=0}^{\infty} n^q \frac{\sin^p (a + ng)}{(b + nh)!} r^n,$$

the author applies the operator $(r\, d/dr)^q$ to S_1, where $(r\, d/dr)^q$ stands for the repetition q times of the operator rd/dr, r and d/dr not being permutable, and where S_1 is of the same form as S except that n^q is wanting. Next S_1 is reduced to

$$S_2 = \sum_{n=0}^{\infty} \frac{\rho^n}{(b + nh)!},$$

where $\rho = re^{(p-2k)\sigma i}$ $(k = 0, 1, 2, \cdots, p)$; and an expression for S_2 is obtained by introducing a function $f(n)$ defined by the formula

$$f(n) = \frac{x^n}{(b+n)!}\sum_{k=1}^{h} \theta_k{}^n,$$

where θ_k is one of the kth roots of unity.

3. Professor W. B. Ford: *A disputed point regarding the nature of the continuum.*

In this paper the author considers the question, originally proposed by Du Bois-Reymond, as to whether a decimal whose digits are chosen in an arbitrary manner (as by the throwing of a die) can properly be regarded as a real number. The question forms but a special example of various questions of broader significance which were extensively discussed by Borel, Hadamard, König and others during the decade preceding the war, their conclusions, however, being widely at variance. It is merely desired in the present paper to point out that such numbers must be regarded as acceptable in case one is to regard the familiar cardinal number theory of Cantor, particularly as regards the continuum, as having a proper place in mathematics, since the derivation of Cantor's results clearly depends upon the use of decimals whose digits are assigned entirely at random.

4. Mrs. Mayme I. Logsdon: *The equivalence of pairs of hermitian forms.*

In this paper the author makes a generalization of the theory of elementary divisors for hermitian λ-matrices with special application to pairs of hermitian forms. If two hermitian matrices A and B are equivalent in the sense that there exist non-singular matrices C and D with determinants free of λ such that $B = CAD$, it is shown that there exists a non-singular matrix P with determinant free of λ such that $B = \bar{P}'AP$, where $\bar{P}$ means the matrix formed from P by replacing each element by its conjugate imaginary and P' means P transposed. It is found that in the case of hermitian forms as well as bilinear and quadratic forms the coincidence of the elementary divisors is a necessary and sufficient condition for equivalence.

The Weierstrass reduction is found to hold where one of the forms is definite, thereby insuring reality of all the elementary

divisors. When complex elementary divisors are present they enter in conjugate pairs. In this case it was found possible to regularize the matrix with respect to the two conjugate linear factors simultaneously and to expand simultaneously with respect to these two factors the terms representing the adjoint form. A simple canonical form was obtained.

5. Mr. C. C. MacDuffee (introduced by Professor L. E. Dickson): *Invariants and vector covariants of linear algebras without the associative law.*

A vector covariant of a linear algebra has been defined by Professor Hazlett (TRANSACTIONS, vol. 19) as a covariant which involves one or more of the units $e_1, \cdots, e_n$. In this paper the author considers the general linear algebra in which the commutative and associative laws of multiplication are not assumed, and shows that, after a vector covariant of weight μ has been expressed linearly in the units, the coefficients of the units are transformed cogrediently, apart from the factor $D^{-\mu}$, with the coefficients of the general number of the algebra under a linear transformation of determinant D. Therefore the coefficients of a vector covariant may be inserted in place of the variables in the covariants of the characteristic equations to give additional relative invariants. After defining a determinant whose elements are hypercomplex numbers for which multiplication is not assumed to be commutative or associative, and after showing that certain elementary theorems concerning determinants apply to these hypercomplex determinants, it is proved that the hypercomplex determinant each of whose n rows is $e_1, e_2, \cdots, e_n$ is a vector covariant of weight 1 for every manner of grouping the factors in each term.

6. Professor E. J. Wilczynski: *Some projective generalizations of geodesics.*

The two most important properties of geodesics are: (*a*) their connection, as extremal curves, with the length of arc integral; (*b*) the property that the osculating plane at any point on such a curve contains the corresponding surface normal.

In this paper, the author considers the generalizations which arise from these two points of view; (*a*) by replacing the length of arc integral by some other integral invariant of the same

general form; (*b*) by replacing the congruence of normals by some other congruence. The two classes of curves obtained in this way do not coincide; but both are included in a larger class of two-parameter curves defined by a second order differential equation of the form

$$\frac{d^2v}{du^2} = M\left(\frac{dv}{du}\right)^3 + N\left(\frac{dv}{du}\right)^2 + P\frac{dv}{du} + Q,$$

where M, N, P, and Q are functions of u and v, the parameters of a surface point. These classes of curves may be characterized by properties of the osculating planes of the curves of a family which pass through a given surface point and dualistically related properties. It then remains to locate the two more special classes mentioned above within the larger class.

7. Professor W. L. Hart: *Summable infinite determinants.*

It has been customary to define the value of an infinite determinant (1) $D = |a_{ij}|$ $(i, j = 1, 2, \cdots)$ by the equation $D = \lim_{n=\infty} D_n$, in case the limit exists, where D_n is the determinant of order n formed by the elements of D for $i, j = 1, 2, \cdots, n$. Utilizing Cesàro summability of order r, the author assigns the value D to the determinant (1) if the sequence $(D_n; n = 1, 2, \cdots)$ is summable to the value D. The results of the paper are restricted to the case $r = 1$. If in (1) we choose all elements as zero except for $a_{2k-1,\,2k} = 1$ and $a_{2k,\,2k-1} = -1$ $(k = 1, 2, \cdots)$, D is summable to the value $\frac{1}{2}$. If quantities c_{ij} are added to the elements of this determinant, then, in case $\Sigma|c_{ij}|$ converges, the new determinant thus obtained is summable and possesses most of the useful properties of normal infinite determinants. There is also exhibited a more complicated type of summable determinants, related in a simple way to a determinant (1), summable to the rational value p/q $(p < q)$, all of whose elements are either $+1$, -1, or 0. It should be noted that no cases in the solution of infinite systems of linear equations can be solved by means of summable determinants which are not also solvable by means of normal infinite determinants.

8. Professor Henry Blumberg: *New properties of all functions.*

For the sake of simplicity the following statements will be made to refer to functions $z = f(x, y)$ of two variables. A point (ξ, η, ζ), where $\zeta = f(\xi, \eta)$, of the surface $z = f(x, y)$

is said to be *densely approached* if for every $\gamma > 0$ two numbers $\alpha > 0$ and $\beta > 0$ exist such that the projections upon the XY-plane of the surface points in the rectangular parallelopiped $\xi - \alpha \leqq x \leqq \xi + \alpha$, $\eta - \beta \leqq y \leqq \eta + \beta$, $\zeta - \gamma \leqq z \leqq \zeta + \gamma$ are everywhere dense in the rectangle $\xi - \alpha \leqq x \leqq \xi + \alpha$, $\eta - \beta \leqq y \leqq \eta + \beta$. The author has found that the points (x, y) for which (x, y, z) is densely approached constitute a residual set (set of second category, according to Baire) of the XY-plane; conversely, for every residual set R, there exists a function $f(x, y)$ such that (x, y, z) is densely approached if and only if (x, y) is in R. It follows that for every function $f(x, y)$ a dense set D of the XY-plane exists, such that f is continuous if it is defined only in D. The results flow from a general theorem concerning *closed region-point* relations. There are generalizations for n dimensions, function space and more general spaces.

9. Professor E. B. Van Vleck: *On non-loxodromic substitution groups in n dimensions.*

This paper is a continuation of a previous paper. For linear substitutions $z' = (az + b)/(cz + d)$ without a common pole there exist two types of non-loxodromic groups, the one consisting of non-loxodromic substitutions having a common invariant circle, while the other contains exclusively elliptic substitutions which are without a common invariant circle. On the author's definition of a non-loxodromic substitution in n dimensions corresponding results hold, but the analog of the elliptic group exists only in space of an odd number of dimensions.

10. Professor G. A. Miller: *An overlooked infinite system of groups of order pq^2, p and q being prime numbers.*

Lists of the possible abstract groups of order pq^2, p and q being prime numbers, were published almost simultaneously by Cole and Glover in the AMERICAN JOURNAL OF MATHEMATICS, vol. 15 (1893), and by O. Hölder in the MATHEMATISCHE ANNALEN, vol. 43 (1893). In a subsequent article published in volume 46 of the latter journal, Hölder directed attention to the fact that the enumeration of these groups contained in the first article mentioned above was incomplete. In the present article the author directs attention to the fact that Hölder's enumeration is also incomplete and observes

that the same incompleteness appears in the enumerations found in the first and also in the second edition of W. Burnside's *Theory of Groups of Finite Order* (1897 and 1911), as well as elsewhere.

There is a system which contains $q - 1$ distinct abstract groups for a given value of p and q, while only one of these groups is found in the published lists.

11. Professor L. E. Dickson: *Fallacies and misconceptions in diophantine analysis.*

This paper has appeared in the April number of this BULLETIN.

12. Professor L. E. Dickson: *A new method in diophantine analysis.*

This paper has appeared in the May number of this BULLETIN.

13. Professor T. H. Hildebrandt: *On a general theory of functions.* (Preliminary report.)

In recent years there have been a number of attempts in the direction of extending the results of the theory of functions of n variables to other cases, that of infinitely many variables, and functionals of continuous functions having received the most attention. This work has been done chiefly by Fréchet, Hart and Gateaux. In this paper an attempt is made to extend by generalization the domain of application of the results already attained, and derive new ones in the general domain. The author uses the basis recently suggested by Lamson (AMERICAN JOURNAL OF MATHEMATICS, vol. 42 (1920), p. 245). So far there have been obtained theorems of mean value, a theorem on implicit functions, existence theorems of differential equations, and some theorems on analytic functions.

14. Professor Arnold Dresden: *Some new formulas in combinatory analysis.*

In this paper the author presents some formulas involving binomial coefficients, which proved of importance in work on symmetric functions with which he is occupied. A search of the literature of combinatory analysis has failed to bring these formulas to light. They are therefore believed to be new.

15. Professor J. B. Shaw: *Generational definition of linear associative hypernumbers.*

Let the n^2 hypernumbers λ_{ij} $(i, j = 1, \cdots, n)$ and certain nilpotent hypernumbers θ_{rs} be given; then the general linear associative hypernumber is of the form

$$\Sigma\lambda_{ij}A_{ij}(\theta_{aj})B_{ijk}(\theta_{bk})C_{ijkl}(\theta_{cl}) \cdots.$$

The laws by which the λ's and the θ's may be permuted in a product are discussed in this paper. They are essential to the algebra to which the general hypernumber belongs.

16. Professor J. B. Shaw: *On Hamiltonian products. Second paper.**

Our range is the totality of functions of a single real variable such that the product of any two is integrable (Lebesgue, Stieltjes, etc.) over a given interval. Each function is represented with a different argument save as specified. Let $\alpha_1(s_1), \alpha_2(s_2), \cdots$ be functions of the totality considered. Then there are two processes considered to be fundamental. The first consists in constructing out of n functions a function of n arguments as follows. Write the product of the n functions $\alpha_1(s_1)\alpha_2(s_2)\cdots\alpha_n(s_n)$. Form similar products from this by permuting the arguments only into every possible arrangement, giving the result a sign $+$ or $-$ according to the inverions, as in determinants. The results are then added. This is represented by $A_n\alpha_1\alpha_2\cdots\alpha_n$.

The second fundamental process consists in starting from the product $\alpha_1(s_1)\alpha_2(s_2)\cdots\alpha_n(s_n)$ as before, but now forming products by making two arguments identical in every possible manner, the sign of the term being determined by the number of inversions necessary to bring the two functions containing the identical arguments together. Each term is then integrated as to the equal arguments over the given interval. If we add the results, we obtain a function of $n-2$ arguments indicated by $A_{n-2}\alpha_1\alpha_2\cdots\alpha_n$. Repeating the process, or what is the same thing, making two pairs of arguments identical in every possible way, and then integrating as to each of the two, determining signs as before, leads to the function $A_{n-4}\alpha_1\alpha_2\cdots\alpha_n$. Then we define the Hamiltonian product of the functions $\alpha_1, \alpha_2, \cdots, \alpha_n$ as the function

* See this BULLETIN, (2), vol. 16 (1910), p. 304.

$$H[\alpha_1(s_1), \alpha_2(s_2), \cdots, \alpha_n(s_n)] = A_n\alpha_1\alpha_2\cdots\alpha_n + A_{n-2}\alpha_1\alpha_2\cdots\alpha_n + \cdots + A_1\alpha_1\alpha_2\cdots\alpha_n \text{ (or } A_0\alpha_1\alpha_2\cdots\alpha_n) \text{ for the last term.}$$

The author considers the resulting theorems.

17. Professor F. E. Wood: *Congruences characterized by certain coincidences.*

The equations of certain lines covariantly related to a general congruence of lines have previously been obtained with respect to various tetrahedra of reference. In this paper the author obtains the equations of these lines with respect to a single fundamental tetrahedron of the congruence as well as the general equations which transform the equations of any configuration related to either of two other tetrahedra of reference into the equations of the same configuration related to this fundamental tetrahedron.

The possible coincidences of these lines, for every line of the congruences, are studied in detail. The congruences characterized by certain coincidences are obtained in a canonical form. In particular, the class of congruences for which the directrix of the first kind of each focal sheet coincides with the directrix of the second kind of the other sheet is found to be the same as the class of congruences associated with functions of a complex variable by Wilczynski.

18. Professor E. P. Lane: *A general theory of congruences.*

Geometers who have considered conjugate nets from the point of view of projective differential geometry have for the most part been content to study a single net, together with its attendant configurations, and have been forced to carry out a tedious transformation whenever it has been desired to consider another conjugate net on the same surface.

The author lays the foundation for a theory of all the conjugate nets on a surface, using the following method. The asymptotic net is taken as parametric. Then the fundamental invariants and covariants of an arbitrary conjugate net are calculated in terms of the asymptotic parameters. The resulting formulas are comparatively simple, and their usefulness is demonstrated by applications to pencils of nets, harmonic nets, isothermally conjugate nets, and plane nets.

Among other theorems, it is proved that if there are more than three harmonic nets in one pencil, the pencil is an isother-

mally conjugate quadratic pencil. And if there are exactly three harmonic nets in a pencil, the pencil is the Segre-Darboux pencil. It is shown that the envelope of the ray of a point for all nets of a pencil is a conic.

19. Professor John Eiesland: *The group of motions of an Einstein space.*

The question to what extent an Einstein space is determined by its group of motions seems to be of interest both from a geometrical and from a physical standpoint. Three assumptions L, M, and N have been made. L is of group-theoretical nature and postulates that the general space $\Sigma_1^4 g_{ik} dx_i dx_k$ admits a certain 4-parameter intransitive group of motions which keeps the origin of the space $t = const.$ fixed. The general space admitting the group G_4 is shown to be of the form

$$(1) \quad -ds^2 = (1 + \phi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) - 2\varphi_3 dRdt - \varphi_1 dt^2,$$

where φ_1, φ_2, φ_3 are functions of R. The static space, for which $\varphi_3 = 0$, or,

$$(2) \quad -ds^2 = (1 + \varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) - \varphi^1 dt^2,$$

is then considered. Adopting two additional assumptions, M and N, we arrive at Schwarzschild's form. The two assumptions are as follows. M. The discriminant of (2) is invariant and equal to $R^4 \sin^2(\theta - c^2)$. (This assumption is also known as Kottler's fundamental hypothesis.*) N. The sum of the principal Riemannian curvatures of the space

$$ds_0^2 = (1 + \varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

is equal to zero.

By varying the last assumption other forms of approximately Euclidean spaces have been obtained. Thus, if the sum is inversely proportional to the fourth power of R, we obtain Weyl's line-element for the case of a spherical mass with a constant electrical charge. In the final part of the paper is considered a gravitational space admitting a two-parameter group of motions, namely a translation along the t-axis and a rotation about the z-axis.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

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* F. Kottler, *Grundlagen der Einsteinschen Gravitationstheorie*, ANNALEN DER PHYSIK, (4), vol. 56, p. 409.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

The thirty-seventh regular meeting of the San Francisco Section of the American Mathematical Society was held at Stanford University, on Saturday, April 9. The chairman of the Section, Professor D. N. Lehmer, presided. The total attendance was twenty-eight, including the following fourteen members of the Society:

Professor R. E. Allardice, Professor B. A. Bernstein, Professor H. F. Blichfeldt, Professor Florian Cajori, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. F. R. Morris, Professor C. A. Noble, Professor T. M. Putnam, Dr. S. E. Urner, Dr. A. R. Williams.

After the regular programme, interesting papers were read by Professor D. L. Webster, of Stanford University, on the quantum theory, and by Dr. T. L. Kelly, also of Stanford University, on the new theory of dispersion.

The dates of the next two regular meetings of the Section were fixed as October 22, 1921, and April 8, 1922.

Titles and abstracts of the papers read at this meeting follow below. Professor Bell's papers were read by title.

1. Professor Florian Cajori: *Euclid of Alexandria and the bust of Euclid of Megara.*

Professor Cajori proves that the portrait issued by the Open Court Publishing Company as representing Euclid of Alexandria, the geometrician, is in fact a portrait of Euclid of Megara. This article appeared in SCIENCE for April 29, 1921.

2. Professor Florian Cajori: *The spread of the Newtonian and the Leibnizian notations of the calculus.*

This paper appears in the present number of this BULLETIN.

3. Professor H. F. Blichfeldt: *The approximate solution in integers of a set of linear equations.*

Consider the n functions $f_1 = |x_1 + \alpha_1 z + a_1|$, $\cdots$, $f_n = |x_n + \alpha_n z + a_n|$, involving $n + 1$ variables x_1, $\cdots$, x_n, z and $2n$ given constants $\alpha_1, \cdots, \alpha_n, a_1, \cdots, a_n$, the

numbers $\alpha_1, \cdots, \alpha_n$ being irrationals which satisfy no equation of the form $k_1\alpha_1 + \cdots + k_n\alpha_n + k = 0$, where $k, k_1, \cdots, k_n$ are integers not all zero. By a theorem of Kronecker it follows that, if a positive quantity ϵ as small as we please is given, then a set of integers $x_1, \cdots, x_n, z$ exist for which the functions f_i are all $< \epsilon$. It is proved in this paper that, given an arbitrary function $\varphi(z)$ of a positive integer z, provided that it represents a positive number approaching infinity with z and that $\varphi(z+1) \geqq \varphi(z)$, a set of irrationals $\alpha_1, \cdots, \alpha_n$ may be defined, restricted as above, such that the condition $F < 1/\varphi(z)$ is satisfied by only a finite number of sets of integers $x_1, \cdots, x_n, z$; here F represents any one of the elementary symmetric functions of degree not higher than $n-1$ of the n functions f_i. On the other hand, whenever the set of irrationals $\alpha_1, \cdots, \alpha_n$ is given, restricted as above, a set of integers $x_1, \cdots, x_n, z$ exists for which every $f_i < \epsilon$, and at the same time $\Pi_{i=1}^n f_i < N/z$, where N depends on n only.

4. Professor M. W. Haskell: *Autopolar curves and surfaces.* In this paper, the author gives the following method for deriving all autopolar curves and surfaces. There is a (1, 1) correspondence between the line-elements of a given curve and the line-elements of its polar reciprocal with respect to a given conic. Every pair of corresponding line-elements determines a pair of line-elements belonging to the given conic. The locus of the foci of the involution defined by these two pairs of line-elements is a curve which is autopolar with respect to the given conic. Every autopolar curve with respect to the given conic can be derived in this way. Curves and surfaces autopolar with respect to a given quadric surface can be derived in the same way, substituting surface-elements for line elements.

The author then gives illustrations of multiply auto-polar curves: a special quintic with five cusps autopolar with respect to six conics, and a special quintic with three cusps and three conjugate points autopolar with respect to four conics.

5. Mr. P. H. Daus (introduced by Professor Lehmer): *Normal ternary continued fractions.*

The author discusses an extension of Jacobi's ternary continued fraction algorithm, and points out certain similarities between it and the ordinary continued fraction expansion.

6. Mr. D. V. Steed (introduced by Professor Lehmer): *The hyperspace generalization of the lines on the cubic surface.*

In this paper it is shown that a necessary condition for the existence of a finite number of linear spaces of dimensions d on the general hypersurface of order r in space of n dimensions is that r, n, and d satisfy the equation

$$\binom{r+d}{d} = (n-d)(d+1).$$

In particular the case where $d = 1$ is considered, and a method for the enumeration of the lines on the general hypersurface of order $2n - 3$ in space of n dimensions is developed. The numbers found for space of dimensions four, five, six, and seven are 2875; 698,005; 306,142,821; and 211,039,426,895, respectively, but by means of the recursion formulas given it is possible to make the corresponding enumeration for space of any dimension.

7. Professor D. N. Lehmer: *On the computation of interest on certain kinds of investments.*

The equations of high degree which are often involved in the computation of the rate of interest on annuities, when treated by Newton's method, yield certain correction formulas which avail to compute the rate to as high a degree of accuracy as may be desired and which require of the computer no more knowledge of the theory of equations or of algebra than is necessary to substitute quantities in a formula. The author exhibits these correction formulas for the rate of interest on a bond, bought at a certain price, and on an annuity whose present value, or whose amount, is given.

The paper is to appear in the AMERICAN JOURNAL OF ACCOUNTANCY.

8. Professor E. T. Bell: *The Bernoullian functions occurring in the arithmetical applications of elliptic functions.*

In applying elliptic functions to theorems on divisors, Glaisher, Halphen and others have given formulas involving Bernoullian functions.

In this paper it is shown that sixteen distinct Bernoullian functions, and no more, can arise in the arithmetic applications of elliptic theta functions and their quotients. The set is considered in detail with applications.

9. Professor E. T. Bell: *Anharmonic polynomial generalizations of the numbers of Bernoulli and Euler.*

In this paper the author discusses two cyclic sets and one anharmonic set which for every degree n degenerate when $z = 1$ either to Bernoulli or Euler numbers of rank n or to others naturally dependent upon these. The entire subject is developed by means of an extension of the symbolic calculus of Blissard and Lucas, and is shown to be simply isomorphic to the algebra of the twelve Jacobian elliptic functions of Glaisher. This analysis leads to several novel features, among which may be mentioned: the definition and solution of purely symbolic linear difference equations; symbolic addition theorems whereby the polynomials of degree $m + n$ can be easily calculated from those of degrees m, n; and a wholly new interpretation of Kronecker's symmetric functions which he took as the point of departure for his discussion of Bernoulli numbers. It is shown that the polynomials cannot be obtained by linear recurrences, and the appropriate recurrences are derived. Finally congruences to a prime modulus are discussed in detail. The algebraic relations and congruences between the polynomials degenerate for $z = 1$ to recurrences and congruences for Bernoulli and Euler numbers. Some of the degenerate cases were given by Lucas and others, thus affording checks.

10. Professor E. T. Bell: *Note on the prime divisors of the numerators of Bernoulli's numbers.*

This note, which will appear shortly in the AMERICAN MATHEMATICAL MONTHLY, contains a generalization of a result due to John Couch Adams, viz. if p is an odd prime which divides neither $2^r + 1$ nor $2^r - 1$, then the numerator of B_{2pr} (in Lucas' notation) is divisible by p. For $r = 1$ we get Adams' theorem. The proof follows from the series for sn u.

11. Professor E. T. Bell: *Proof of an arithmetic theorem due to Liouville.*

This paper appeared in the March number of this BULLETIN.

12. Professor E. T. Bell: *On a general arithmetic formula of Liouville.*

This paper appeared in the April number of this BULLETIN.

B. A. BERNSTEIN,
Secretary of the Section.

THE GENERAL THEORY OF APPROXIMATION BY POLYNOMIALS AND TRIGONOMETRIC SUMS.

REPORT PRESENTED BEFORE THE AMERICAN MATHEMATICAL SOCIETY AT THE SYMPOSIUM IN CHICAGO ON MARCH 25, 1921.

BY PROFESSOR DUNHAM JACKSON.

1. *Introduction.* A great part of the progress made in mathematical analysis during the last hundred years has been closely related, either logically or historically, to the study of Taylor's and Fourier's series. The power series is fundamental, or can be made fundamental, for the theory of functions of complex variables, and the theory of functions of real variables has been elevated to its present dignity and scope largely by the successive additions made to meet the demands of the inquirer into the properties of trigonometric series.*

One of the outlying portions of the structure is built around the problem of the approximate representation of an arbitrary function by means of polynomials or by means of finite trigonometric sums in general; that is, with the admission of coefficients other than those of a specified number of terms of a particular series. This theory, which has grown to its present extent mainly within twenty years, forms the subject of the following report. It can not be set off from the rest by any sharp line of demarcation, but there are certain well-marked processes of development which it is not difficult to trace. It has seemed expedient for various reasons to make this paper an introduction to the literature of the subject, rather than an independent exposition of any considerable part of the theory. Even with this limitation, the treatment is merely illustrative, not in any sense exhaustive.

2. *Tchebychef's Theory and its Generalizations.* Between 1850 and 1860, Tchebychef † discussed the problem of de-

* Cf. Van Vleck, *The influence of Fourier's series upon the development of mathematics*, SCIENCE, vol. 39 (1914), pp. 113–124.

† Tchebychef 1 (this form of reference will be adopted when there is occasion to cite more than one paper by the same author), *Théorie des mécanismes connus sous le nom de parallélogrammes*, MÉMOIRES PRÉSENTÉS À L'ACADÉMIE IMPÉRIALE DES SCIENCES DE ST.-PÉTERSBOURG PAR DIVERS SAVANTS, vol. 7 (1854), pp. 539–568; *Oeuvres*, vol. 1, Petrograd, 1899, pp. 111–143.

Tchebychef 2, *Sur les questions de minima qui se rattachent à la représenta-*

termining a polynomial $P_n(x)$, of given degree n, to approximate a given continuous function $f(x)$, in such a way that the maximum of the absolute value of the error,

$$\text{Max. } |f(x) - P_n(x)|,$$

shall be as small as possible. His reasoning, as recorded at that early date, was naturally incomplete, according to present standards. It was put into modern form by Kirchberger* in 1902. The idea was then rapidly carried further. It was applied by Fréchet† and by J. W. Young‡ to problems of a considerably higher degree of generality, and in particular to representation by finite trigonometric sums, by Tonelli§ to functions of more than one variable (a phase already touched upon by Kirchberger and Fréchet), and by Sibirani‖ to representation by linear combinations of a given set of linearly independent functions generally; and it has been extended in a variety of other ways.

The most striking facts with regard to the Tchebychef polynomial of given degree n, for a given continuous function $f(x)$, in a given interval $a \leqq x \leqq b$, are that it is uniquely determined, and that the error $f(x) - P_n(x)$ takes on its greatest numerical value not just once, but at least $n + 2$ times, alternately with opposite signs.¶ When $n = 0$, for

tion approximative des fonctions, MÉMOIRES DE L'ACADÉMIE IMPÉRIALE DES SCIENCES DE ST.-PETERSBOURG, (6), SCIENCES MATHÉMATIQUES ET PHYSIQUES, vol. 7, (1859), pp. 199–291; *Oeuvres*, vol. 1, pp. 273–378.

In preparing the present paper, I have consulted only the collected works, from which the citations of the original memoirs are quoted.

* Kirchberger, *Ueber Tchebychefsche Annäherungsmethoden*, Dissertation, Göttingen, 1902. See also Blichfeldt, *Note on the functions of the form* $f(x) \equiv \varphi(x) + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_n$ *which in a given interval differ the least possible from zero*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 2 (1901), pp. 100–102.

† Fréchet 1, *Sur l'approximation des fonctions par des suites trigonométriques limitées*, COMPTES RENDUS, vol. 144 (1907), pp. 124–125; Fréchet 2, *Sur l'approximation des fonctions continues périodiques par les sommes trigonométriques limitées*, ANNALES DE L'ECOLE NORMALE SUPÉRIEURE, (3), vol. 25 (1908), pp. 43–56.

‡ J. W. Young, *General theory of approximation by functions involving a given number of arbitrary parameters*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 8 (1907), pp. 331–344.

§ Tonelli, *I polinomi d'approssimazione di Tchebychev*, ANNALI DI MATEMATICA, (3), vol. 15 (1908), pp. 47–119.

‖ Sibirani, *Sulla rappresentazione approssimata delle funzioni*, ANNALI DI MATEMATICA, (3), vol. 16 (1909), pp. 203–221.

¶ In the paper of 1854, the earliest reference to the subject with which I am acquainted, Tchebychef says (*Oeuvres*, vol. 1, p. 114):

"Soit fx une fonction donnée, U un polynome du degré n avec des coefficients arbitraires. Si l'on choisit ces coefficients de manière à ce que la

example, it is evident that the best constant is that midway between the greatest and least values of $f(x)$, so that the maximum deviation is reached at least twice, once positively and once negatively. A little experimentation with a ruler and a curve drawn on paper will leave little doubt as to the correctness of the statement for $n = 1$, and perhaps will suggest at the same time the idea of the general proof. The latter, however, is by no means trivial, and it is a very satisfactory exercise in the application of elementary theorems of algebra and of the simplest principles of analysis.

For the corresponding trigonometric problem, let it be supposed that $f(x)$ is of period 2π, and is continuous for all real values of x. Among all expressions of the form

$$T_n(x) = a_0 + a_1 \cos x + \cdots + a_n \cos nx \\ + b_1 \sin x + \cdots + b_n \sin nx,$$

that is, among all finite trigonometric sums of order n, there will be one and just one for which the maximum of the quantity $|f(x) - T_n(x)|$ has the smallest possible value. This particular sum is characterized by the fact that the quantity $f(x) - T_n(x)$ reaches its greatest numerical value at least $2n + 2$ times in a period, alternately with opposite signs. It will be seen that the number $2n + 2$ here, like the number $n + 2$ in the polynomial case, is one more than the number of arbitrary coefficients in question.

Except for certain specific references in a later section, there is perhaps no occasion to dwell upon the subject longer here, since its main features are very readably presented in standard treatises.* The explicit determination of the approximating function, or of the degree of approximation attained, is extraordinarily difficult, even in relatively simple cases, because the dependence of the approximating function on $f(x)$ is not linear. It is especially noteworthy that the Tchebychef

différence $fx - U$, depuis $x = a - h$, jusqu'à $x = a + h$, reste dans les limites les plus rapprochées de 0, la différence $fx - U$ jouira, *comme on le sait*, de cette propriété:

'Parmi les valeurs les plus grandes et les plus petites de la différence $fx - U$ entre les limites $x = a - h$, $x = a + h$, on trouve au moins $n + 2$ fois la même valeur numérique.'"

The italics are mine. Kirchberger, op. cit., p. 6, states that the problem was originally proposed by Poncelet.

* Cf., e.g., Borel, *Leçons sur les Fonctions de Variables Réelles et les Développements en Séries de Polynomes*, Paris, 1905, pp. 82–92; de la Vallée Poussin 1, *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, Paris, 1919, see chapters 6, 7.

theory did not lead in any direct way to a recognition of the fundamental fact brought out in the next section, that there exist uniformly convergent processes of approximation by polynomials or by finite trigonometric sums, for an arbitrary continuous function $f(x)$, although the approximating functions of Tchebychef by definition yield the most rapidly convergent of all such processes.

3. *Weierstrass's Theorem.* Let $f(x)$ be an arbitrary continuous function over the interval $a \leqq x \leqq b$, and let ϵ be an arbitrary positive quantity. Then there will always exist a polynomial $P(x)$ such that $|f(x) - P(x)| < \epsilon$ throughout the interval. This theorem was first published by Weierstrass* in 1885. Almost at the same time, and independently of Weierstrass, Runge,† without formulating this particular conclusion, supplied the materials for a second proof, quite different in form. Other writers successively added other demonstrations, in great variety.‡ A time came when there was no longer any distinction in inventing a proof of Weierstrass's theorem, unless the new method could be shown to possess some specific excellence, in the way of simplicity, for example, or in rapidity of convergence. The question of degree of convergence will be the principal concern of the remainder of the paper. In respect to simplicity, mention may be made particularly of the proof of Lebesgue,§ which depends on the application of the binomial theorem to the representation of a function whose graph is a broken line, the proof of Landau,|| which makes use of an especially con-

* Weierstrass 1, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*, BERLINER SITZUNGSBERICHTE, 1885, pp. 633–639.

† Runge, *Über die Darstellung willkürlicher Functionen*, ACTA MATHEMATICA, vol. 7 (1885), pp. 387–392, together with an earlier paper by the same author: *Zur Theorie der eindeutigen analytischen Functionen*, ACTA MATHEMATICA, vol. 6 (1885), pp. 229–244; pp. 236–237. The first-named paper deals with the approximation of an arbitrary continuous function by means of a rational function, while the other supplies the necessary facts about the approximate representation of rational functions by means of polynomials.

‡ Cf., e.g., Borel, op. cit., pp. 50–61.

§ Lebesgue 1, *Sur l'approximation des fonctions*, BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 22 (1898), pp. 278–287; cf. de la Vallée Poussin 1, pp. 3–5.

|| Landau, *Über die Approximation einer stetigen Funktion durch eine ganze rationale Funktion*, RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO, vol. 25 (1908), pp. 337–345.

venient form of (so-called *singular*) definite integral, and Simon's modification* of Landau's proof, in which the definite integral is replaced by a finite sum.

The trigonometric form of the theorem, to the effect that an arbitrary continuous function of period 2π can be uniformly represented by a finite trigonometric sum with any assigned degree of accuracy, was established by Weierstrass himself, not in the first paper already cited, but immediately afterwards.† The polynomial and trigonometric cases are not only susceptible of parallel treatment, but are readily convertible one into the other by a simple change of variable. Among the direct proofs for the trigonometric case may be mentioned, for simplicity and elegance, those of de la Vallée Poussin,‡ using a definite integral, and Kryloff,§ using the finite sum which corresponds to de la Vallée Poussin's integral.

4. *First Studies of Degree of Convergence. De la Vallée Poussin's Problem.* It has long been known, in more or less detail, that there is a relation between the properties of continuity of a function and the degree of accuracy that can be attained in its approximate representation by specified means. For example, Picard,|| in his *Traité d'Analyse*, points out incidentally that if $f(x)$ is a function of period 2π possessing a kth derivative which is (essentially) of limited variation, the coefficient of cos nx or sin nx in the Fourier series for $f(x)$ does not exceed a constant multiple of $1/n^{k+1}$ in absolute value. This leads to a theorem about the degree of approximation to $f(x)$ given by the partial sum of its Fourier series, that is by a particular trigonometric sum of the nth order.

Lebesgue,¶ in 1908, formally proposed the problem of discussing the relation between the accuracy of polynomial ap-

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*Simon, *A formula of polynomial interpolation*, ANNALS OF MATHEMATICS, (2), vol. 19 (1918), pp. 242–245.

† Weierstrass 2, same title as his first paper, BERLINER SITZUNGSBERICHTE, 1885, pp. 789–805.

‡ de la Vallée Poussin 2, *Sur l'approximation des fonctions d'une variable réelle et de leurs dérivées par des polynômes et des suites limitées de Fourier*, BULLETINS DE L'ACADÉMIE ROYALE DE BELGIQUE, Classe des Sciences, 1908, pp. 193–254.

§ Kryloff, *Sur quelques formules d'interpolation généralisée*, BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 41 (1917), pp. 309–320.

|| Picard, *Traité d'Analyse*, 2nd ed., vol. 1, pp. 252–253, 255–256.

¶ Lebesgue 2, *Sur la représentation approchée des fonctions*, RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO, vol. 26 (1908), pp. 325–328.

proximation and the requisite degree of the polynomial, and indicated that if $f(x)$ satisfies a Lipschitz condition for $a \leqq x \leqq b$, it can be approximately represented by a polynomial of the nth degree with an error not exceeding a constant multiple of $\sqrt{(\log n)/n}$. This result was improved by de la Vallée Poussin,* who obtained the limit $1/\sqrt{n}$ in place of that just mentioned.† For a more restricted class of functions, retaining enough generality to include any function whose graph is a broken line with a finite number of segments, he found‡ the still closer limit $1/n$. Then he added,§

"Il serait très intéressant de savoir s'il est impossible de représenter l'ordonnée d'une ligne polygonale avec une approximation d'ordre supérieur à $1 : n$ par un polynôme de degré n."

This remark has been the direct or indirect occasion of most of the subsequent work on the subject.

5. *Inner Limit of Approximation. S. Bernstein's Theory.* The simplest example of a function coming within the specifications of the problem is the function $|x|$, considered in the interval from -1 to 1. It was shown by de la Vallée Poussin‖ that the maximum error of an approximating polynomial of degree n can not approach zero faster than $1/(n \log^3 n)$. S. Bernstein¶ and the writer** independently replaced this limit by $1/(n \log n)$. The final solution, verifying de la Vallée Poussin's surmise that $1/n$ is the actual limit, was given by S. Bernstein,†† in a notable prize essay for the Belgian Academy,

* de la Vallée Poussin 2, already cited, p. 222.

† The hypothesis was somewhat differently stated by de la Vallée Poussin, but his analysis is immediately applicable to the case of a Lipschitz condition; cf. D. Jackson 1, cited below, p. 10, footnote.

‡ de la Vallée Poussin 3, *Note sur l'approximation par un polynôme d'une fonction dont la dérivée est à variation bornée*, Bulletins de l'Académie royale de Belgique, Classe des Sciences, 1908, pp. 403–410.

§ Footnote, p. 403.

‖ de la Vallée Poussin 4, *Sur les polynomes d'approximation et la représentation approchée d'un angle*, Bulletins de l'Académie royale de Belgique, Classe des Sciences, 1910, pp. 808–844.

¶ S. Bernstein 1, *Sur l'approximation des fonctions continues par des polynomes*, Comptes Rendus, vol. 152 (1911), pp. 502–504.

** D. Jackson 1, *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung*, Dissertation, Göttingen, 1911; see pp. 49–52.

†† S. Bernstein 2, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Mémoire couronné, Brussels, 1912. (Included in Mémoires publiés par la Classe des sciences de

with de la Vallée Poussin's problem for its text. Expanding into a discussion of a wide range of questions in the theory of polynomial approximation generally, the essay contains a number of results comparable in interest with the accomplishment of its original purpose. One of these, by reason of its simplicity and its far-reaching consequences, may well be regarded as one of the most remarkable theorems of recent times. It will be worth while to dwell on it at some length. Here, again, it is possible to speak either of polynomials or of trigonometric sums. The statement is more striking in terms of the latter. For the trigonometric case, the theorem is as follows:

Let $T_n(x)$ be an arbitrary trigonometric sum of order n, and let L be the maximum of its absolute value. Then the maximum of the absolute value of the derivative $T_n'(x)$ can not exceed nL.

For this formulation, to be sure, the credit is not undivided. Bernstein's own statement* asserts merely that $|T_n'(x)|$ can not attain the value $2nL$; his proof is far from simple; and its validity has been called in question,† though this last remark applies only to his discussion of the trigonometric theorem, not to the polynomial case, in which he was primarily interested. The theorem was approached by subsequent writers from different angles, and was finally revealed in its true simplicity by de la Vallée Poussin,‡ whose demonstration is well worth repeating here.

An equivalent form of the assertion to be proved is as follows:

If the maximum of $|T_n'(x)|$ is 1, the maximum of $|T_n(x)|$ can not be less than $1/n$.

Suppose the maximum of $|T_n(x)|$ were less than $1/n$. Then, for any value of the constant C, the function

$$R_n(x) = \frac{1}{n} \sin (nx + C) - T_n(x)$$

would have the sign of $\sin (nx + C)$ at each of the $2n$ points

L'ACADÉMIE ROYALE DE BELGIQUE, (2), vol. 4; the detailed citations below, however, refer to the page-numbers of the essay itself, as printed separately.)

* S. Bernstein 2, p. 20.

† Cf. de la Vallée Poussin, passages cited in next footnote.

‡ de la Vallée Poussin 5, *Sur le maximum du module de la dérivée d'une expression trigonométrique d'ordre et de module bornés*, COMPTES RENDUS, vol. 166 (1918), pp. 843–846; de la Vallée Poussin 1, pp. 39–42.

at which $|\sin(nx + C)|$ has a maximum, in an interval of length 2π. Because of the alternation of these signs, and the periodicity of the functions involved, $R_n(x)$ would then vanish for at least $2n$ distinct values of x in a period, and consequently, by Rolle's theorem, the same thing would be true of the derivative

$$R_n'(x) = \cos(nx + C) - T_n'(x).$$

But if C is chosen so as to make $\cos(nx + C)$ coincide in value with $T_n'(x)$ at a point where the latter is equal to ± 1, $R_n'(x)$ will have a double root at this point. Hence it will have, in an entire period, roots of aggregate multiplicity at least $2n + 1$. For a trigonometric sum of order n, which can not vanish identically—since $R_n(x)$, by reason of its changes of sign, can not be a constant—this is impossible, and the contradiction proves the theorem. It can be shown further,* though not quite so simply, that the maximum of $|T_n(x)|$ will be *greater* than $1/n$, unless $T_n(x)$ has precisely the form $(1/n)\sin(nx + C)$, that is, the latter is the only function for which the limit specified in the theorem is attained.

The corresponding theorem for polynomials may be stated as follows, for a particular interval:

Let $P_n(x)$ be an arbitrary polynomial of degree n, and let L be the maximum of its absolute value for $-1 \leqq x \leqq 1$. Then the maximum of $|\sqrt{1 - x^2}\, P_n'(x)|$ can not exceed nL in the interval specified.

Bernstein established this fact first,† and went from it to the trigonometric case. The relative simplicity of the opposite procedure, on the basis of de la Vallée Poussin's trigonometric proof, has been pointed out by the writer.‡

The most immediate application of the theorem is to a type of argument of which the following is a simple case. Let $f(x)$ be a continuous function of period 2π, and let it be supposed that there exists, for every positive integral value of n, a trigonometric sum $T_n(x)$, of order n, so that

$$|f(x) - T_n(x)| \leqq Q/n^2,$$

where Q is independent of n and x. Then $f(x)$ is the sum of

* Cf. citations in preceding footnote.

† S. Bernstein 2, pp. 6–11.

‡ D. Jackson 2, *On the convergence of certain trigonometric and polynomial approximations*, Transactions of the American Math. Society, vol. 22 (1921), pp. 158–166; see p. 162.

the uniformly convergent series

$$T_1 + (T_2 - T_1) + (T_4 - T_2) + (T_8 - T_4) + \cdots,$$

or, if $f(x) - T_n(x) = r_n(x)$,

$$(1) \quad f(x) = T_1 + (r_1 - r_2) + (r_2 - r_4) + (r_4 - r_8) + \cdots.$$

It will be shown that if the series on the right is differentiated term by term, the resulting series will be uniformly convergent, and $f(x)$ consequently must have a continuous first derivative. Since $|r_n| \leqq Q/n^2$, by hypothesis,

$$|r_1 - r_2| \leqq Q + \frac{Q}{2^2} < 2Q, \quad |r_2 - r_4| \leqq \frac{Q}{2^2} + \frac{Q}{4^2} < \frac{2Q}{2^2},$$
$$|r_4 - r_8| < \frac{2Q}{4^2}, \cdots.$$

But since $r_1 - r_2$ is a trigonometric sum of order 2, etc., it follows from the theorem on the derivative of a trigonometric sum that

$$|(r_1 - r_2)'| < 2 \cdot 2Q, \quad |(r_2 - r_4)'| < 4 \cdot \frac{2Q}{2^2}, \cdots,$$

and, generally, that the derivative of the $(p+1)$th term on the right in (1) does not exceed

$$2^p \cdot \frac{2Q}{2^{2(p-1)}} = \frac{Q}{2^{p-3}}.$$

The last quantity is the general term of a convergent series, and the uniform convergence of the series of derivatives is established. It is evident that this illustration by no means exhausts the possibilities of the method. Thus we may state the following more general theorem.*

If $f(x)$ can be represented by trigonometric sums of order n with an error not exceeding $Q/n^{k+\alpha}$, where k is a positive integer or zero, and $0 < \alpha < 1$, then $f(x)$ has a continuous kth derivative satisfying a Lipschitz condition of order α, that is

$$|f^{(k)}(x') - f^{(k)}(x'')| \leqq \lambda |x' - x''|^\alpha,$$

where λ is a constant.† The idea can be carried still further.

* Cf. de la Vallée Poussin 1, p. 57; also S. Bernstein 1, and S. Bernstein 2, pp. 22–23, 27.

† The value $\alpha = 1$ is ruled out in the statement of the theorem; but if the hypothesis is satisfied for $\alpha = 1$, as in the preceding illustration ($k = 1$), it is of course satisfied, and the conclusion holds, for an arbitrary value of $\alpha < 1$.

The theorem concerning the order of approximation to $|x|$, like the theorem on the derivative of a trigonometric sum, is most accessible at present through the exposition of de la Vallée Poussin.* The idea of his proof is as follows. In the first place, the problem is referred to the equivalent one of representing $|\sin x|$ by a trigonometric sum of order n in x. Lebesgue† had pointed out that no trigonometric sum of order n can give an error smaller than the corresponding remainder in the Fourier series, multiplied by a quantity of the order of $1/\log n$. As it is readily recognized that the error in the Fourier series for $|\sin x|$ is of the order of $1/n$, the quantity $1/(n \log n)$, already mentioned as a step toward the final result, is seen at once to be an inner limit for the order of the best approximation. If the Fejér mean could be used in the same way as the Fourier sum proper, the desired limit $1/n$ would be obtained with equal ease; for Lebesgue's remark is a consequence of the fact that the partial sum of the Fourier series for an arbitrary function can not exceed the maximum of the absolute value of the function itself, multiplied by a quantity of the order of $\log n$, and in the case of the corresponding Fejér mean, this multiplier is replaced by 1. But the method fails at first, because it is essential also that the Fourier sum of order n for a function $T_n(x)$ is identical with $T_n(x)$, while this is not true of the Fejér mean. However, de la Vallée Poussin observes that if $\tau_k(x)$ is the Fejér mean of order $k - 1$, $T_n(x)$ is reproduced identically by the formula

$$T_n(x) = 2\tau_{2n}(x) - \tau_n(x);$$

and from this relation he is able to draw the desired inference, not quite immediately, but by easy steps, in the course of two or three pages. The general theorem which intervenes is as follows.

Let $f(x)$ be a given arbitrary function, let $S_k(x)$ be the partial sum of its Fourier series to terms of order k, and let $T_n(x)$ be any trigonometric sum of order n. Then the maximum of $|f(x) - T_n(x)|$ can not be less than one-fourth the maximum of

$$\left| f(x) - \frac{S_n + S_{n+1} + \cdots + S_{2n-1}}{n} \right|.$$

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* de la Vallée Poussin 6, *Sur la meilleure approximation des fonctions d'une variable réelle par des expressions d'ordre donné*, Comptes Rendus, vol. 166 (1918), pp. 799–802; de la Vallée Poussin 1, pp. 33–37.

† Cf. e.g. Lebesgue 3, *Sur les intégrales singulières*, Annales de la Faculté de Toulouse, (3), vol. 1 (1909), pp. 25–117; see pp. 116–117.

Bernstein himself, in the prize essay, proves the theorem about the polynomial representation of $|x|$ in two different ways.* Both of his demonstrations depend on an extension, interesting in itself, of the Tchebychef theory to the problem of the approximate representation of a given continuous function by linear combinations of powers of x with arbitrarily given exponents.† The exponents may be any positive real numbers, not necessarily integral. There is a real generalization, however, even if they are merely selected integers; there would be no point in saying that some powers *may* be omitted, in the approximating polynomials used, for that is always understood, as a matter of course; but the problem is essentially changed if it is demanded that certain powers *shall* be omitted.

By way of obtaining a corresponding generalization of Weierstrass's theorem, Bernstein further inquires under what circumstances a sequence of positive powers x^{α_1}, x^{α_2}, $\cdots$, will be sufficient (in technical language, *complete*) for the uniformly convergent representation of an arbitrary continuous function.‡ He derives a condition which is necessary, and others which are sufficient. Müntz and Szász§ later established a condition which is both necessary and sufficient, namely (except for minor qualifications) the very simple requirement that the series $\Sigma\,(1/\alpha_n)$ diverge. The last-named author extends the discussion to the case of complex exponents.

The rest of Bernstein's memoir must be dismissed with the utmost brevity in the present summary. Among the topics treated may be mentioned some results concerning the absolute value of a polynomial for complex values of the argument,‖ a study of the approximate representation of analytic functions of a complex variable,¶ theorems with regard to the

* S. Bernstein 2, pp. 55–62. See also S. Bernstein 3, *Sur la meilleure approximation de $|x|$ par des polynomes de degrés donnés*, ACTA MATHEMATICA, vol. 37 (1914), pp. 1–57.

† S. Bernstein 2, pp. 38–41.

‡ S. Bernstein 2, pp. 78–84.

§ Szász, *Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen*, MATHEMATISCHE ANNALEN, vol. 77 (1916), pp. 482–496. The paper of Müntz, which I have not seen, is cited by Szász, loc. cit., p. 483, footnote.

‖ pp. 13–15.

¶ pp. 36, 65–76, 94–95, and elsewhere. This subject, which has an extensive literature of its own, must be left outside the scope of the present review.

degree of convergence of Fourier's series and the Fejér means of Fourier's series,* in the spirit of the following section, and applications of the theory of approximation to questions of the existence of derivatives of functions of more than one variable.†

A paper by Montel ‡ presents a noteworthy continuation of Bernstein's work in various directions, particularly with reference to derivatives of fractional order.

6. *Outer Limit of Approximation.* From Picard's inequalities, already cited, for the coefficients in a Fourier series, it follows that, if the function developed has a continuous kth derivative which is of limited variation, the remainder after n terms of the series does not exceed a constant multiple of $1/n^k$. For $k = 1$, the same outer limit of error for the approximation obtainable by means of finite trigonometric sums of order n was found by de la Vallée Poussin,§ under a similar but somewhat more general hypothesis. Most often, however, such discussion has involved the hypothesis of a Lipschitz condition, either on the function itself or on one of its derivatives.

For the case of a function satisfying a Lipschitz condition, de la Vallée Poussin,|| as has already been noted, found the order of approximation $1/\sqrt{n}$, and the same limit was recorded a little later by Lebesgue,¶ both for polynomials and for trigonometric sums. From the point of view of a rapid survey of the problem, it is not necessary to specify each time which kind of approximating function is used, as the two classes of results are to a large extent interchangeable.

Another step in advance was presently taken by Lebesgue,** who proved that the remainder in the Fourier series for a function of the kind under discussion can not exceed a constant multiple of $(\log n)/n$.

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* pp. 85–93.

† pp. 97–103.

‡ Montel, *Sur les polynomes d'approximation*, Bulletin de la Société Mathématique de France, vol. 46 (1919), pp. 151–192.

§ de la Vallée Poussin 3.

|| de la Vallée Poussin 2, p. 222.

¶ Lebesgue 3, pp. 112–115.

** Lebesgue 4, *Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz*, Bulletin de la Société Mathématique de France, vol. 38 (1910), pp. 184–210; pp. 199–201.

He also showed that this result is final, as far as the Fourier series is concerned, that is that there exist functions satisfying a Lipschitz condition, for which the remainder does not approach zero any more rapidly.*

There still remained a question as to the degree of approximation that might be attained by a different choice of the approximating functions. The writer showed that a function satisfying a Lipschitz condition can be represented by a trigonometric sum of the nth order,† or by a polynomial of the nth degree,‡ with a maximum error not exceeding a constant multiple of $1/n$, and that the limit $1/n$ in this general statement can not be replaced by any infinitesimal of higher order.§ The truth of the last part of this assertion of course also follows immediately from Bernstein's results with regard to the function $|x|$, which were published a little later. The writer subsequently obtained inequalities for the magnitude of the numerical constants involved, so that the following statements can be made in summary.‖

If $f(x)$ is a function of period 2π satisfying everywhere the condition

$$|f(x') - f(x'')| \leqq \lambda |x' - x''|, \tag{2}$$

it is possible to represent $f(x)$ by a trigonometric sum of order n with a maximum error not exceeding $K\lambda/n$, where K is an absolute constant; the statement is true for $K = 3$, but not for any value of K less than $\pi/2$.

If $f(x)$ satisfies the condition (2) for $a \leqq x \leqq b$, it can be represented throughout this interval by a polynomial of the nth degree with a maximum error not exceeding $L\lambda(b - a)/n$, where L is an absolute constant; the statement is true for $L = 1\frac{1}{2}$, but not for any value of L less than $\frac{1}{2}$.

The upper values for K and L, as actually computed by the writer, were slightly less than 3 and $1\frac{1}{2}$ respectively, and they can be further reduced by other methods, due particularly to

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* Lebesgue 4, pp. 202–206.

† D. Jackson 1, Theorem VI; see also top of p. 46.

‡ D. Jackson 1, Theorems I, IV.

§ D. Jackson 1, Theorems XII, XIII.

‖ D. Jackson 3, *On approximation by trigonometric sums and polynomials*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 13 (1912), pp. 491–515; Theorems I, II, VI, VII. There is a misprint in the statement of Theorem VII; for L_2 should be read L_1.

Gronwall.* The ultimate determination of the best values for K and L, however, is still an open question.

The order of approximation being once established for the case of a Lipschitz condition, corresponding results can be deduced without difficulty both for more general and for more restrictive hypotheses. By way of generalization,† let $f(x)$ be an arbitrary continuous function, of period 2π, say, to restrict attention to the trigonometric case, and let $\omega(\delta)$ be the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. Let $f_1(x)$ be a function represented graphically by a broken line, equal to $f(x)$ at the points $x = 2j\pi/n$ ($j = 0, \pm 1, \pm 2, \cdots$), and varying linearly from one of these points to the next. Then, for any x, $f(x)$ and $f_1(x)$ will each differ from the value at the nearest vertex point by not more than $\omega(2\pi/n)$, and

$$|f(x) - f_1(x)| \leqq 2\omega(2\pi/n)$$

for all values of x. Since $f_1(x)$ satisfies a Lipschitz condition with coefficient $\omega(2\pi/n)/(2\pi/n)$, there will exist trigonometric sums $T_n(x)$, of order n, so that

$$|f_1(x) - T_n(x)| \leqq 3\cdot \frac{\omega(2\pi/n)}{2\pi/n}\cdot\frac{1}{n} = \frac{3}{2\pi}\,\omega(2\pi/n),$$

and

$$|f(x) - T_n(x)| \leqq \left(\frac{3}{2\pi} + 2\right)\omega\left(\frac{2\pi}{n}\right)\cdot$$

A similar result can also be obtained directly by means of a definite integral, without the intermediate reference to the Lipschitz condition.‡

On the other hand, suppose that $f(x)$, once more assumed to

* Unpublished letters to the present writer, January 21 and February 16, 1913; *On approximation by trigonometric sums*, this BULLETIN, vol. 21 (1914–15), pp. 9–14. See also de la Vallée Poussin 1, pp. 44–46. My computation, in the Transactions article cited, involved the ratio of two integrals, J_m' and J_m; within a few weeks after its publication, I was in receipt of letters from Messrs. Gronwall, I. Schur, and D. Cauer (the last-named, then a student at Göttingen, writing informally in behalf of Professor Landau), which contained, among other data, two proofs that

$$J_m = \frac{\pi}{3}\left(1 + \frac{1}{2m^2}\right),$$

and seven proofs (including one quoted by Schur from Marcel Riesz) that $\lim_{m\to\infty} J_m' = \log 2$.

† See, e.g., D. Jackson 1, Theorem VIII. The fundamental idea of the proof is derived from Lebesgue 4, p. 202.

‡ Cf. de la Vallée Poussin 1, pp. 44–46.

have the period 2π, is not merely continuous itself, but has a continuous first derivative satisfying the Lipschitz condition

$$|f'(x') - f'(x'')| \leqq \lambda |x' - x''|.$$

There is a finite trigonometric sum $T_n'(x)$ such that

$$|f'(x) - T_n'(x)| \leqq \frac{3\lambda}{n}.$$

Moreover, it follows from the proof of the theorem on which this assertion is based, though not from the statement of the theorem itself, that T_n' can be taken so that its constant term is zero,* and its integral therefore is also a trigonometric sum of order n. Let the expression

$$f(0) + \int_0^x T_n'(x)dx$$

be denoted by $T_n(x)$, and let $f'(x) - T_n'(x) = r_n(x)$. Then

$$f(x) = T_n(x) + \int_0^x r_n(x)dx.$$

Of course the integral on the right does not exceed a constant multiple of $1/n$, but it is possible to say much more than that. For the integral has a derivative which does not exceed $3\lambda/n$, that is the integral itself *satisfies a Lipschitz condition with coefficient* $3\lambda/n$, *and can be represented by a trigonometric sum of order n with an error not exceeding* $9\lambda/n^2$. This sum, combined with $T_n(x)$, gives an equally close approximation for $f(x)$. In general†—though the proof requires a little attention in detail—*if* $f(x)$ *has a* $(k-1)$*th derivative which satisfies a Lipschitz condition with coefficient* λ, *then* $f(x)$ *can be approximately represented by a trigonometric sum of order n with an error not exceeding* $3^k\lambda/n^k$. There is a corresponding but slightly less simple result for polynomial approximation.‡

From the trigonometric theorem it follows immediately, by a remark of Lebesgue already cited, that if $f(x)$ has a $(k-1)$th

* That is, if the function for which an approximate representation is sought, $f'(x)$ in this case, is such that its integral is periodic, the particular approximating function defined in the course of the demonstration will also have a periodic integral.

† D. Jackson 4, *On the approximate representation of an indefinite integral and the degree of convergence of related Fourier's series*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 14 (1913), pp. 343–364; Theorem III. Cf. also D. Jackson 1, Theorem VII, and D. Jackson 3, Theorems III, VIII.

‡ D. Jackson 4, Theorem VII; cf. D. Jackson 1, Theorems II, IVa, and D. Jackson 3, Theorems IV, IX.

derivative satisfying a Lipschitz condition, the remainder after terms of the nth order in the Fourier series for $f(x)$ does not exceed $(\log n)/n^k$, multiplied by a quantity independent of n and x. It is noteworthy that a different method yields a still closer result here, to the extent that the constant multiplier obtained is *independent of k*. If the coefficient in the Lipschitz condition is λ, the error does not exceed $\lambda(\log n)/n^k$, multiplied by an absolute constant.*

Finally, we may note a case in which one of the theorems of the preceding section has an exact converse.† *If $f(x)$ has a continuous kth derivative satisfying a Lipschitz condition of order α, where k is a positive integer or zero, and $0 < \alpha < 1$, then $f(x)$ can be represented by trigonometric sums of order n with an error not exceeding $Q/n^{k+\alpha}$, where Q is independent of n and x.* Taken without regard to its converse, the statement can be made both more precise, by exhibiting the dependence of Q on the coefficient in the Lipschitz condition, and more general, by varying the hypothesis on the kth derivative.

7. *Trigonometric Interpolation.* In the theory which forms the subject of § 6, much use is made of formulas involving definite integrals. It is possible to vary the treatment by replacing the integrals by finite sums. From this substitution, which works out most satisfactorily in the trigonometric case, results an extensive theory of trigonometric interpolation.‡ The ordinary formula of interpolation with equidistant ordinates has of course been known for a long time, but its

* Cf. S. Bernstein 2, pp. 92–93; D. Jackson 4, Theorem X; de la Vallée Poussin 1, pp. 23–25, 27–29.

† Cf. de la Vallée Poussin 1, pp. 51–52, 57; also D. Jackson 4, Theorem IV.

‡ Cf., e. g., de la Vallée Poussin 7, *Sur la convergence des formules d'interpolation entre ordonnées équidistantes*, Bulletins de l'Académie royale de Belgique, Classe des Sciences, 1908, pp. 319–403; Faber 1, *Über stetige Funktionen*, Mathematische Annalen, vol. 69 (1910), pp. 372–443, see pp. 417–443; Faber 2, *Über die interpolatorische Darstellung stetiger Funktionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 192–210; D. Jackson 5, *On the accuracy of trigonometric interpolation*, Transactions of the American Math. Society, vol. 14 (1913), pp. 453–461; D. Jackson 6, *A formula of trigonometric interpolation*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 371–375; D. Jackson 7, *Note on trigonometric interpolation*, Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915), pp. 230–232; D. Jackson 8, *On the order of magnitude of the coefficients in trigonometric interpolation*, Transactions of the American Math. Society, vol. 21 (1920), pp. 321–332.

convergence seems to have been first studied in detail by de la Vallée Poussin and Faber within fifteen years. The convergence properties of the interpolating formula are similar to, but not absolutely identical with, those of the Fourier series, which it formally resembles. The analogy extends to questions of degree of convergence, and to the various methods of trigonometric approximation which have been devised for one special purpose or another.

8. *The Method of Least mth Powers.* Of modest importance in itself, perhaps, but of some interest as affording a field for the application of the preceding results, is the theory of what may be called the method of least mth powers.* If $f(x)$ is a continuous function of period 2π (to restrict attention to the trigonometric case once more), it is well known that the trigonometric sum $T_n(x)$, of order n, for which the integral

$$\int_0^{2\pi} [f(x) - T_n(x)]^2 dx$$

has the least possible value, is the partial sum of the Fourier series for $f(x)$. The condition determining $T_n(x)$ can be generalized by writing, in place of the square of the error, the mth power of its absolute value, where m is an arbitrary positive exponent, most conveniently assumed to be greater than 1. The resulting sum $T_n(x)$ is not readily amenable to calculation, because, like the Tchebychef sum, it does not depend linearly on $f(x)$; but it is possible to show that it always exists, is uniquely determined, for $m > 1$, and approaches the Tchebychef sum as a limit when m becomes infinite. Furthermore, if m is held fast and n is allowed to increase indefinitely, $T_n(x)$ will converge uniformly to the value $f(x)$, under suitable hypotheses. The proof depends on Bernstein's theorem concerning the derivative of a trigonometric sum, and on the possibility of representing $f(x)$ by a properly chosen sum with a specified degree of approximation.

THE UNIVERSITY OF MINNESOTA.

* Cf. D. Jackson 9, *On functions of closest approximation*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 22 (1921), pp. 117–128, and D. Jackson 2, already cited. The polynomial case had previously been treated by Pólya, in a note with which I was not acquainted at the time of writing my own papers: *Sur un algorithme, etc.*, COMPTES RENDUS, vol. 157 (1913), pp. 840–843.

THE EINSTEIN SOLAR FIELD.

BY PROFESSOR LUTHER PFAHLER EISENHART.

(Read before the American Mathematical Society April 23, 1921.)

The Schwarzschild form of the linear element of the Einstein field of gravitation of a mass m at rest with respect to the space-time frame of reference is

$$(1) \qquad ds^2 = \left(1 - \frac{2m}{u_1}\right) dt^2 - \frac{u_1}{u_1 - 2m} du_1^2 - u_1^2(du_2^2 + \sin^2 u_2\, du_3^2),$$

where t is the coordinate of time, and u_1, u_2, u_3 are space coordinates. Since the coefficients in (1) are independent of t, the particle moves in the 3-space S_3 whose linear element is

$$(2) \qquad ds_0^2 = \frac{u_1}{u_1 - 2m} du_1^2 + u_1^2(du_2^2 + \sin^2 u_2\, du_3^2).$$

If we put

$$(3) \qquad \begin{aligned} x_1 &= u_1 \sin u_2 \cos u_3, & x_2 &= u_1 \sin u_2 \sin u_3, \\ x_3 &= u_1 \cos u_2, & x_4 &= 4m\sqrt{\frac{u_1}{2m} - 1}, \end{aligned}$$

we have

$$(4) \qquad ds_0^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

Hence S_3 is immersed in the euclidean space of four dimensions, S_4, whose rectangular coordinates are x_i $(i = 1, \cdots, 4)$.* Moreover, as follows from (3), S_3 is the quartic variety defined by

$$(5) \qquad x_1^2 + x_2^2 + x_3^2 = \left(\frac{x_4^2}{8m} + 2m\right).$$

We shall show that equations (3) define the only three-spread with the linear element (2) in euclidean four-space by making use of the following theorem of Bianchi:† In euclidean n-space $(n > 3)$ every hypersurface is not deformable unless at least $n - 2$ of the principal radii of curvature are infinite.

* Cf. Kasner, AMERICAN JOURNAL OF MATHEMATICS, vol. 43, p. 132 (April, 1921).

† *Lezioni*, vol. 1, p. 467.

In fact we show that all of the principal radii of curvature of (5) are finite. If X_i denote the direction-cosines of the normal to (5), that is

$$\sum_{i=1}^{4} X_i \frac{\partial x_i}{\partial u_j} = 0 \ (j = 1, 2, 3), \qquad \sum X_i^2 = 1,$$

then

$$X_1 = \sqrt{\frac{2m}{u_1}} \sin u_2 \cos u_3, \qquad X_2 = \sqrt{\frac{2m}{u_1}} \sin u_2 \sin u_3,$$

$$X_3 = \sqrt{\frac{2m}{u_1}} \cos u_2, \qquad X_4 = -\sqrt{1 - \frac{2m}{u_1}}.$$

If we define functions Ω_{rs} by

$$\Omega_{rs} = -\sum_i \frac{\partial X_i}{\partial u_r} \frac{\partial x_i}{\partial u_s},$$

we find $\Omega_{rs} = 0$ $(r \neq s)$ and

$$\Omega_{11} = \frac{1}{u_1 - 2m}\sqrt{\frac{m}{2u_1}}, \quad \Omega_{22} = -\sqrt{2mu_1}, \quad \Omega_{33} = -\sqrt{2mu_1}\sin^2 u_2.$$

The principal radii of curvature are given by*

$$\frac{1}{R_1} = \frac{\Omega_{11}}{\dfrac{u_1}{u_1 - 2m}} = \sqrt{\frac{m}{2u_1^3}}, \qquad \frac{1}{R_2} = \frac{\Omega_{22}}{u_1^2} = -\sqrt{\frac{2m}{u_1^3}},$$

$$\frac{1}{R_3} = \frac{\Omega_{33}}{u_1^2 \sin^2 u_2} = \sqrt{\frac{2m}{u_1^3}},$$

which are finite since $u_1 \neq 0$.

In accordance with the Einstein theory the world-line of a particle in the gravitational field is a geodesic of the space with the linear element (1), that is a curve along which $\int ds$ is stationary; and the world-line of a ray of light is a curve for which $ds = 0$ and $\int dt$ is stationary. In each case the frame of reference can be so chosen that a particular path satisfies the condition $u_2 = \pi/2$.† From (3) it follows that for this path $x_3 = 0$, and hence the path considered by astronomers is the projection upon the plane $x_3 = 0$ of a curve on the surface

$$x_1^2 + x_2^2 = \left(\frac{x_4^2}{8m} + 2m\right)^2.$$

* Bianchi, l.c., pp. 368, 472.

† Cf. Eddington, *Report on the Relativity Theory of Gravitation*, p. 49.

This is the surface of revolution of a parabola of latus rectum $8m$ about its directrix. A similar result was obtained by Flamm* who considered the surface, in euclidean three-space, for which the linear element is given by (2) for $u_2 = \pi/2$.

PRINCETON UNIVERSITY,
April 16, 1921.

A COVARIANT OF THREE CIRCLES.

BY PROFESSOR A. B. COBLE.

(Read before the American Mathematical Society April 23, 1921.)

Dr. J. L. Walsh † has stated the following theorem.

THEOREM. *If the double ratio,* $(z_1, z_3 \mid z_2, z)$, *of the four points* z_1, z_2, z_3, z *in the complex plane is a real number* λ, *then as the points* z_1, z_2, z_3 *run over the circles* C_1, C_2, C_3 (*and their interiors*) *respectively, the locus of* z *is a circle* (*and its interior*).

This locus is evidently a covariant, under the inversive group, of the three given circles, which is rational in λ. We find in (8) its equation and incidentally prove the theorem.

In conjugate coordinates $z, \bar{z}$, a circle is

$$C_1(z) = a_1 z\bar{z} + \alpha_1 z + \bar{\alpha}_1 \bar{z} + b_1 = 0,$$

where a_1, b_1 are real, and α_1, $\bar{\alpha}_1$ are conjugate imaginary. The bilinear invariant of two circles $C_1(z)$, $C_2(z)$ is

$$[C_1, C_2] = \alpha_1\bar{\alpha}_2 + \alpha_2\bar{\alpha}_1 - a_1 b_2 - a_2 b_1.$$

It vanishes when the two circles are orthogonal. When they coincide it becomes $[C_1 C_1] = 2(\alpha_1\bar{\alpha}_1 - a_1 b_1)$. This vanishes when C_1 is a *point circle*, i.e. one whose equation is

$$(1) \qquad P_{z_i}(z) = (z - z_i)(\bar{z} - \bar{z}_i) = 0.$$

It is easily verified that

$$[C_1, P_{z_i}(z)] = -C_1(z_i); \quad [P_{z_i}(z), P_{z_k}(z)] = -P_{z_i}(z_k) = -P_{z_k}(z_i).$$

The two point circles of the pencil $C(z) + \mu K(z) = 0$ are determined by

$$[C + \mu K, C + \mu K] = [C, C] + 2\mu\, [CK] + \mu^2\, [KK] = 0.$$

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* PHYSIK. ZEITSCHR., vol. 17 (1916), p. 449.

† TRANSACTIONS AMER. MATH. SOCIETY, vol. 22 (1921), p. 101. The geometric proof of this theorem given by Dr. Walsh is very complicated. The method of proof followed here is considered by Dr. Walsh (loc. cit.,

They coincide and the circles $C(z)$, $K(z)$ touch when

$$[KC]^2 - [KK]\,[CC] = 0.$$

We begin the proof with the condition

$$\frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)} = \lambda, \tag{2}$$

and set

$$l_1 = (z_2 - z_3)(z - z_1), \quad l_2 = (z_3 - z_1)(z - z_2), \quad l_3 = (z_1 - z_2)(z - z_3),$$

$$l_1 + l_2 + l_3 = 0; \tag{3}$$

$$q_1 = \lambda(\lambda - 1), \qquad q_2 = \lambda, \qquad q_3 = 1 - \lambda,$$

$$q_2q_3 + q_3q_1 + q_1q_2 = 0. \tag{4}$$

The condition (2), or $(\lambda + l_3/l_1) = 0$, when multiplied by its conjugate, $(\lambda + \bar{l}_3/\bar{l}_1)$, is easily reduced by the use of (3) and (4) to the symmetrical form $q_1l_1\bar{l}_1 + q_2l_2\bar{l}_2 + q_3l_3\bar{l}_3 = 0$. Again, in the notation of (1), this condition is

$$q_1P_{z_2}(z_3)\cdot P_{z_1}(z) + q_2P_{z_3}(z_1)\cdot P_{z_2}(z) + q_3P_{z_1}(z_2)\cdot P_{z_3}(z) = 0.$$

For fixed values of λ, z_2, z_3, z this is the equation which determines z_1. If z_1 lies on a circle $C_1(z) = 0$, then

$$K \equiv q_1P_{z_2}(z_3)\cdot C_1(z) + q_2C_1(z_3)\cdot P_{z_2}(z) + q_3C_1(z_2)\cdot P_{z_3}(z) = 0.$$

For fixed z_2, z_3, and z_1 variable in the circle C_1, $K(z) = 0$ is the equation of the circle within which z lies. Now let z_2 range from its fixed position outward in all directions toward the boundary of a circle C_2. Then the circle $K(z)$ ranges outward in all directions from its original position toward the boundary of an envelope which is the *outer part* of the envelope of the ring of circles $K(z)$ as z_2 runs over the circumference of the circle C_2. This envelope is the locus of points z for which *K regarded as a circle in the variables* z_2, $\bar{z}_2$ touches the given circle C_2 and therefore the equation of the envelope is

$$[KC_2]^2 - [KK][C_2C_2] = 0.$$

We shall show that $[KK]$ is a perfect square and therefore the envelope factors into a pair of circles of which we want the outer. We notice that $K(z_2)$ breaks up into three terms, $K_1(z_2) + K_2(z_2) + K_3(z_2)$. Hence

$$[KK] = \Sigma[K_iK_i] + 2\Sigma[K_iK_j] \quad (i, j = 1, 2, 3;\ i \neq j).$$

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footnote, p. 102) but rejected because of algebraic difficulties. These however are not inherent. The algebraic method has, moreover, the decided advantage of furnishing the required envelope in covariant form.

To within the coefficients q the terms $[K_iK_j]$ are all alike and equal to $-P_{z_3}(z)\cdot C_1(z)\cdot C_1(z_3)$. But according to (4) the sum of the coefficients of these three terms vanishes. Moreover $[K_iK_i]=0$ $(i=1,2)$ since $K_1(z_2)$ and $K_2(z_2)$ are point circles. Hence $[KK]=q_3^2\cdot(P_{z}^{z}(z))^2\cdot[C_1C_1]$. Also

$$[KC_2] = -q_1C_1(z)\cdot C_2(z_3) - q_2C_1(z_3)\cdot C_2(z) + q_3P_{z_3}(z)\cdot[C_1C_2].$$

Thus for proper choice of the sign of the radicals the outer part of the envelope is the circle

$$(5)\quad L \equiv -q_1C_1(z)\cdot C_2(z_3) - q_2C_1(z_3)\cdot C_2(z) + q_3P_{z_3}(z)\{[C_1C_2] - \sqrt{[C_1C_1]}\sqrt{[C_2C_2]}\}.$$

We now let z_3 run over a circle $C_3(z)=0$. As before the envelope of the circle $L(z)$ in (5) is the tact-invariant of L regarded as a circle in the variable z_3 and of $C_3(z_3)=0$. It is therefore

$$(6)\qquad [LC_3]^2 - [LL][C_3C_3] = 0.$$

Again the term $[LL]$ is a perfect square. In fact

$$[LL] = q_1^2C_1^2(z)\cdot[C_2C_2] + q_2^2C_2^2(z)\cdot[C_1C_1] + 2q_1q_2C_1(z)\cdot C_2(z)\cdot[C_1C_2] + 2q_3(q_1+q_2)C_1(z)\cdot C_2(z)\{[C_1C_2] - \sqrt{[C_1C_1]}\sqrt{[C_2C_2]}\},$$

the term in q_3^2 dropping out since q_3 is the coefficient of a point circle. Since $q_3(q_1+q_2) = -q_1q_2$ this becomes

$$(7)\qquad [LL] = \{q_1C_1(z)\sqrt{[C_2C_2]} + q_2C_2(z)\sqrt{[C_1C_1]}\}^2.$$

Hence the final envelope (6) factors into two circles (necessarily inner and outer) and the theorem is proved.

In order to obtain the equation of the envelope we note that

$$[LC_3] = -q_1C_1(z)\cdot[C_2C_3] - q_2C_2(z)\cdot[C_1C_3] - q_3C_3(z)\cdot\{[C_1C_2] - \sqrt{[C_1C_1]}\sqrt{[C_2C_2]}\}.$$

This, together with (7), yields the factors of (6), whence

The locus of z referred to in the theorem is, in explicit form,

$$(8)\quad \begin{aligned}&\lambda(\lambda-1)\cdot C_1(z)\{[C_2C_3] - \sqrt{[C_2C_2]}\sqrt{[C_3C_3]}\}\\ &+ \lambda\cdot C_2(z)\{[C_3C_1] - \sqrt{[C_3C_3]}\sqrt{[C_1C_1]}\}\\ &+ (1-\lambda)\cdot C_3(z)\{[C_1C_2] - \sqrt{[C_1C_1]}\sqrt{[C_2C_2]}\} = 0;\end{aligned}$$

where the sign of the radical $\sqrt{[C_iC_i]}$ is to be taken opposite the sign of the quadratic q_i for given λ.

On account of the symmetry and homogeneity of this result the verification of sign can be made for (5) and $a_1 = a_2 = 1$. We have in (5) two circles which determine the pencil

$$(9)\quad \mu\{-q_1C_2(z_3)\cdot C_1(z) - q_2C_1(z_3)\cdot C_2(z) + q_3P_{z_3}(z)\cdot[C_1C_2]\} - \{q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}P_{z_3}(z)\} = 0.$$

Let C_1, C_2 be small circles around the points z_1, z_2 respectively which themselves are not on a line with the point z_3. The pencil (9) contains the point circle $P_{z_3}(z)$ and therefore another point circle P interior to the two point circles (5) since $P_{z_3}(z)$ is exterior to them. Hence in (9) we must have for $\mu = 0$ the point circle z_3; for $\mu = \mu_R$ the radical axis; for $\mu = 1$, the outer circle (5); for $\mu = -1$ the inner circle (5); and for $\mu = c(-1 < c < 0)$, the point circle P. Thus μ_R, the parameter of the radical axis of the pencil (9) must be positive. But

$$\mu_R = q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}/(-q_1C_2(z_3) - q_2C_1(z_3) + q_3[C_1C_2]).$$

If now the circles C_1, C_2 approach the points z_1, z_2 as limits the denominator of μ_R approaches as a limit

$$(10)\qquad -(q_1\alpha^2 + q_2\beta^2 + q_3\gamma^2)$$

where α, β, γ are the lengths of the sides opposite the vertices z_1, z_2, z_3 of the triangle z_1, z_2, z_3. In terms of λ (10) becomes

$$(11)\qquad -\alpha^2\lambda^2 + (\gamma^2 + \alpha^2 - \beta^2)\lambda - \gamma^2.$$

The discriminant of (11) is

$$(\alpha + \beta + \gamma)(-\alpha + \beta + \gamma)(-\beta + \alpha + \gamma)(\gamma - \alpha - \beta)$$

which is negative. Hence (11) is a definite quadratic form evidently negative for sufficiently large λ. Then (10) is negative for all real values of λ and this requires that $q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}$ be negative. Since $q_1q_2q_3 = -\lambda^2(\lambda - 1)^2$ is negative for all real values of λ, the three radicals must take the same signs as, or opposite signs to, the three quadratics q,

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ON SKEW PARABOLAS.

BY DR. MARY F. CURTIS.

The theorem that a real rectifiable skew parabola is a helix, proved in my note in this BULLETIN, November, 1918, for skew parabolas which can be represented in rectangular coordinates by equations of the form:

$$(1)\qquad x_1 = at,\qquad x_2 = bt^2,\qquad x_3 = ct^3,\qquad abc \neq 0,$$

was extended by Professor Hayashi in this BULLETIN, November, 1919, to cover all real skew parabolas, whose equations he reduces to the form

(2) $x_1 = a_1t^3 + a_3t, \quad x_2 = b_1t^3 + b_2t^2, \quad x_3 = c_1t^3, \quad a_3b_2c_1 \neq 0.$

Professor Loria, in an extract from a letter to Professor D. E. Smith published in this BULLETIN, February, 1921, states that Professor Hayashi's work was unnecessary, in that every (real) skew parabola can be represented in rectangular coordinates by equations of the form (1).

When the coordinates are oblique, (1) does represent every real skew parabola.* Professor Loria's statement that this is equally true when the coordinates are rectangular is, however, at fault. A skew parabola C:

(3) $$x_i = a_it^3 + b_it^2 + c_it + d_i \qquad (i = 1, 2, 3)$$

lies on a unique parabolic cylinder S and meets the ruling R on which the vertices of the orthogonal sections of S lie in a unique finite point P. If $P : t = 0$ is taken as the origin of coordinates, R as the x_3-axis, and the axis of the orthogonal section through P as the x_2-axis, equations (3) become

(4) $x_1 = c_1t, \quad x_2 = b_2t^2, \quad x_3 = a_3t^3 + b_3t^2 + c_3t, \quad a_3b_2c_1 \neq 0,$

and these are as simple equations representing a general skew parabola in rectangular coordinates as can be found.†

It is clear that the osculating plane at $P : t = 0$ is the plane $x_3 = 0$ if and only if $b_3 = c_3 = 0$. Then (4) reduces to (1).

THEOREM. *A real skew parabola C can be represented in rectangular coordinates by equations of the form* (1) *if and only if, at the point in which C meets the line of vertices of the parabolic cylinder on which C lies, the osculating plane of C is orthogonal to the cylinder.*

The representation (4) for the general skew parabola, the theorem and the representation (1) for the particular skew parabolas which the theorem singles out hold in the complex domain, unless the rulings of the cylinder S are minimal, or the parallel planes each of which cuts S in a single ruling are minimal, or both the rulings and the planes are minimal. In such cases simple representations of the skew parabola exist.

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* ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, III, C 2: O. Staude. *Flächen 2. Ordnung und ihre Systeme*, p. 234. ENCYCLOPÉDIE DES SCIENCES MATHÉMATIQUES III, 22 (1914), p. 130.

† O. Staude: *Analytische Geometrie der Kubischen Kegelschnitte* (1913), p. 139. The equations (2) which refer the curve to the tangent, principal normal and binormal at an arbitrary point may, for some purposes, be preferable to the equations (4).

THE DISPERSION OF OBSERVATIONS.

BY PROFESSOR JULIAN LOWELL COOLIDGE.

In the study of statistics bearing upon groups of objects, it is of fundamental importance to know whether, and to what extent, the different objects in the same group are comparable, and also to what extent one group is comparable with another. The first writer to study such questions seems to have been Lexis* and subsequent writers have developed his ideas of normal, supernormal and subnormal dispersion, or of Bernoulli, Lexis, and Poisson series.

In all articles dealing with these topics, which have come to the writer's attention, the data are either restricted to series of probabilities or frequency ratios, or else it is assumed that variations are distributed according to the Gaussian law of error. This restriction is unnecessary and unfortunate. The purpose of the present paper is to show how any series of sets of observations may be tested for the normality of their dispersion. The mathematical means employed are of the most elementary nature.

Suppose that we have n observed values $y_1, y_2, \cdots, y_n$. The expression

$$\sqrt{\frac{1}{n}\left[\sum_i\left(y_i-\sum_j\frac{y_j}{n}\right)^2\right]}$$

is called the *dispersion* or *standard deviation* of the set, and is of first importance in the theory of errors of observation, and of statistics. Let us find the mean value of its square. For simplicity, the mean value of y_i shall be called a_i while the mean value of its square is A_i. We shall write the averages,

$$(1/n)\sum y_i = y, \qquad (1/n)\sum a_i = a.$$

The square of the dispersion is

$$(1/n)\sum_i(y_i-y)^2 = (1/n)\sum_i\left\{\Big((y_i-a_i)-(y-a)\Big)+[a_i-a]\right\}^2.$$

In the large round parenthesis, if the averages y and a be replaced by their expanded values, we have the sum of a number of terms, each of which has the mean value 0, and as these terms are independent, the mean value of the product

* *Zur Theorie der Massenerscheinungen in der menschlichen Gesellschaft*, Freiburg, 1877.

of two is also 0. The mean value of the square of the round bracket is the sum of the mean values of the squares of its individual terms. The mean value of the product of the round and square brackets is 0, and the mean value of the square of the square bracket is its ostensible value.

$$(y_i - a_i) - (y - a) = y_i - a_i - \frac{\sum(y_i - a_i)}{n}$$

$$= \frac{(n-1)}{n}(y_i - a_i) - \frac{(y_j - a_j) + (y_k - a_k) + \cdots (y_n - a_n)}{n}.$$

The mean value of $(y_i - a_i)^2$ is $A_i - a_i^2$, and the coefficient will be $(n - 1/n)^2$ in one case, and $1/n^2$ in $(n - 1)$ other cases. Summing, we reach the following theorem.

FUNDAMENTAL DISPERSION THEOREM. *If n independent quantities $y_1, y_2, \cdots, y_n$ be given, their mean values being $a_1, a_2, \cdots, a_n$, while the mean values of the squares are $A_1, A_2, \cdots, A_n$ respectively, and if $y = (1/n)\Sigma_i y_i$, $a = (1/n)\Sigma_i a_i$, then the mean value of the square of the dispersion is*

$$\frac{1}{n}\left[\frac{n-1}{n}\sum_i (A_i - a_i^2) + \sum_i (a_i - a)^2\right].$$

In practice it is convenient, first to replace $(n - 1)/n$ by 1, and, secondly, to replace the mean value of the square of the dispersion by its observed value. We thus obtain the approximate equation

$$(1) \qquad \sum_i (y_i - y)^2 = \sum_i (A_i - a_i^2) + \sum_i (a_i - a)^2.$$

Let us show the practical application of this equation. Suppose that we have N sets, each of s observations,

$$\begin{array}{ll} x_{11}\ x_{12}\ \cdots\ x_{1s}; & \sum_j x_{1j} = x_1, \\ x_{21}\ x_{22}\ \cdots\ x_{2s}; & \sum_j x_{2j} = x_2, \\ \cdots\cdots & \cdots\cdots \\ x_{N1}\ x_{N2}\ \cdots\ x_{Ns}; & \sum_j x_{Nj} = x_N. \end{array}$$

Let the mean value of x_{ij} be a_{ij}, while the mean value of its square is A_{ij}; and let $\Sigma_j a_{ij} = a_i$, $\Sigma_i x_i = Nx$, $\Sigma_i a_i = Na$. Then, by (1), we have

$$\sum_j \left(x_{ij} - \frac{x_i}{s}\right)^2 = \sum_j (A_{ij} - a_{ij}^2) + \sum_j \left(a_{ij} - \frac{a_i}{s}\right)^2.$$

Summing again, we find

$$(2) \qquad \sum_{i,j} \left(x_{ij} - \frac{x_i}{s}\right)^2 = \sum_{i,j} (A_{ij} - a_{ij}^2) + \sum_{i,j} \left(a_{ij} - \frac{a_i}{s}\right)^2.$$

Again, we may write $(x_i - a_i) = \Sigma_j(x_{ij} - a_{ij})$. The mean value of $(x_i - a_i)^2$ is $\Sigma_j(A_{ij} - a_{ij}^2)$. Hence applying (1) once more, we find

$$(3)\qquad \sum_i (x_i - x)^2 = \sum_{i,j} (A_{ij} - a_{ij}^2) + \sum_i (a_i - a)^2.$$

Eliminating $\Sigma_{ij}(A_{ij} - a_{ij}^2)$ between (2) and (3), we obtain

$$(4)\quad \sum_{i,j}\left[\left(x_{ij}-\frac{x_i}{s}\right)^2 - \left(a_{ij}-\frac{a_i}{s}\right)^2\right] = \sum_i \left[(x_i-x)^2-(a_i-a)^2\right].$$

This is our fundamental equation. In practice it is usual to see whether a given series of sets of observations belong to one of three recognized types:

(A) Bernoulli series. Here all of the observations are supposed to be upon the same object, or at least the mean value does not vary from one object to another. We have

$$a_{ij} \equiv a_i,\quad a_i \equiv a,\quad \sum_{i,j}\left(x_{ij} - \frac{x_i}{s}\right)^2 \doteq \sum_i (x_i - x)^2.$$

Such a series is said to have *normal dispersion.*

(B) Lexis series. All observations in one set are supposed to bear on the same object, but the object varies from set to set.

$$a_{ij} \equiv a_i,\quad a_i \not\equiv a,\quad \sum_{i,j}\left(x_{ij} - \frac{x_i}{s}\right)^2 < \sum_i (x_i - x)^2.$$

Such a series is said to have *supernormal dispersion.*

(C) Poisson series. The objects within a set are supposed to differ from one another, but all sets are supposed comparable.

$$a_{ij} \not\equiv a_i,\quad a_i \equiv a,\quad \sum_{i,j}\left(x_{ij} - \frac{x_i}{s}\right)^2 > \sum_i (x_i - x)^2.$$

Such a series is said to have *sub-normal dispersion.*

What we can do in practice is this. *We calculate the quantities $\Sigma_{i,j}\,(x_{ij} - x_i/s)^2$ and $\Sigma_i(x_i - x)^2$. If they be virtually equal, we are sure that the members of each set can not all be equal, unless the sets are all the same, and vice versa. If the first be less than the second, the various sets can not be the same. If the first be greater than the second, the individuals must differ within a set.*

It is well, in conclusion, to show how our formulas connect with the usual formulas for the dispersion of ratios. The problem is to see whether certain frequency ratios vary from case to case, or from set to set. If the generic letter for one of our probabilities be p_{ij}, and this represent the chance that

x_{ij} takes the value 1, while in the contrary case it takes the value 0, then a_i/s represents the average probability for the ith set.

$$\sum_j p_{ij} = sp_i = a_i \qquad \sum_i p_i = Np = a/s \qquad p + q = 1.$$

$$A_{ij} - a_{ij}^2 = \text{Mean value of } (x_{ij} - p_{ij})^2 = p_{ij} - p_{ij}^2.$$

From (3), we find

$$\sum_i (x_i - x)^2 = \sum_{i,j} (p_{ij} - p_{ij}^2) + s^2 \sum_i (p_i - p)^2.$$

Now

$$\sum_j (p_{ij} - p_i)^2 = \sum_j p_{ij}^2 - sp_i^2.$$

Hence

$$\sum_j p_{ij}^2 = sp_i^2 + \sum_j (p_{ij} - p_i)^2.$$

Similarly

$$\sum_i p_i^2 = Np^2 + \sum_i (p_i - p)^2,$$

$$\sum_i (x_i - x)^2 = \sum_i [sp_i - sp_i^2 - \sum_j (p_{ij} - p_i)^2 + s^2(p_i - p)^2]$$
$$= Nspq - \sum_{i,j}(p_{ij} - p_i)^2 + (s^2 - s)\sum_i (p_i - p)^2,$$

$$(5) \quad \frac{1}{N}\sum_i (x_i - x)^2 = spq - \frac{1}{N}\sum_{i,j}(p_{ij} - p_i)^2 + \frac{s^2 - s}{N}\sum_i (p_i - p)^2.$$

To take up the particular cases:

(A) BERNOULLI SERIES. The probability is constant throughout.

$$p_{ij} \equiv p_i \equiv p, \quad \frac{1}{N}\sum_i (x_i - x)^2 = spq.$$

(B) LEXIS SERIES. The probability is constant throughout a set, but varies from set to set.

$$p_{ij} \equiv p_i, \quad p_i \not\equiv p. \quad \frac{1}{N}\sum_i (x_i - x)^2 = spq + \frac{s^2 - s}{N}\sum_i (p_i - p)^2.$$

(C) POISSON SERIES. The probability varies within the set, but all sets are comparable.

$$p_{ij} \not\equiv p_i, \quad p_i \equiv p. \quad \frac{1}{N}\sum_i (x_i - x)^2 = spq - \sum_j (p_{ij} - p_i)^2.$$

These are the standard formulas.*

HARVARD UNIVERSITY,
April, 1921.

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* Conf. Fisher, *Mathematical Theory of Probabilities*, New York, 1915, pp. 120 ff.

THE ISOMORPHISMS OF COMPLEX ALGEBRA.

BY DR. NORBERT WIENER.

(Read before the American Mathematical Society December 28, 1917.)

Definitions and Conventions. By the *transform* by a function $\phi(x)$ of a function $F(x_1, x_2, \cdots, x_n)$ we mean the expression

$$\varphi\{F(\varphi^{-1}(x_1), \varphi^{-1}(x_2), \cdots, \varphi^{-1}(x_n))\}.$$

We shall consider ∞ as a possible argument for a function, and we shall define $F(\infty)$ to be $\lim\limits_{x \doteq \infty} F(x)$. Similarly if $\lim\limits_{x = \infty} F(x) = \infty$, as $|x|$ grows without limit in every possible way, we shall say that $F(\infty)$ is ∞, while if $\lim\limits_{x \doteq a} F(x) = \infty$ as x approaches a in every possible way, we shall say that $F(a) = \infty$. Similar conventions will be established for functions of more than one variable.

A function F of the complex variable z is said to be *continuous* at z_1 if $\lim\limits_{|\epsilon|=0} F(z_1 + \epsilon) = F(z_1)$.

A function $F(x_1, x_2, \cdots, x_n)$ will be said to be *continuous in general* if there are only a finite number of sets $(x_1, x_2, \cdots, x_n)$ for which $F(x_1, x_2, \cdots, x_n)$ fails to be continuous.

A variable μ is said to *depend uniquely* on $x_1, x_2, \cdots, x_n$ if there are only a finite number of sets $x_1, x_2, \cdots, x_n$ for which μ is not uniquely determined, and if for these μ is undefined.

THEOREM. *If a function Φ is single-valued, as well as its inverse, over the set of arguments consisting of all complex numbers and ∞, and if it is continuous in general, and transforms every algebraic function into an algebraic function, it is a linear function or its conjugate.* The proof of this will involve almost no considerations except those of elementary algebra.

To begin with, let us consider any function of the form

$$F(x, y) = \frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}$$

which does not degenerate into a constant nor into a function of a single variable. Such a function has the following properties.

(1) $F(x, y)$ depends uniquely on x and y.
(2) x depends uniquely on y and $F(x, y)$.
(3) y depends uniquely on x and $F(x, y)$.

It is easy to show that these properties will also belong to any transform of F by a biunivocal function

Now, no algebraic function not of the form

$$\frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}$$

has properties (1), (2), and (3). Any algebraic function with these properties must be obtained by solving for z an equation $P(x, y, z) = 0$, where P is some polynomial. Since z is uniquely determined by x and y, we may assume, without any real loss of generality, that P is of the form*

$$[g(x, y) + zh(x, y)]^m,$$

which we may write

$$g^m + mg^{m-1}hz + \text{terms in higher powers of } z.$$

Since P is a polynomial, g^m and $g^{m-1}h$ are polynomials. Let us call these Q and R, respectively. P is then of the form

$$\frac{1}{g^{m^2-m}}\{g^m + zhg^{m-1}\}^m = \psi(x, y)\{Q(x, y) + zR(x, y)\}^m,$$

where Q and R may be taken so as to be mutually prime, by removing any common factor and transferring it to the factor ψ. By considerations of symmetry, we may write

$$\begin{aligned} P(x, y, z) &= \psi'(y, z)\{Q'(y, z) + xR'(y, z)\}^{m'} \\ &= \psi''(x, z)\{Q''(x, z) + yR''(x, z)\}^{m''}. \end{aligned}$$

It follows from a consideration of the irreducible factors of P that we may write

$$P(x, y, z) = (\alpha + \beta x + \gamma y + \delta xy \\ - \epsilon z - \xi xz - \eta yz - \vartheta xyz)^m,$$

where $\alpha, \cdots, \vartheta$ are constants. Hence the only algebraic functions satisfying conditions (1), (2), and (3) are of the form

$$F(x, y) = \frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}.$$

We shall now state a lemma which may readily be established by the usual method of undetermined coefficients. The function $1 - x/y$ is the only function of the form

$$F(x, y) = \frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}$$

which satisfies the four conditions

* Notice that this is only true in complex algebra.

(1) $F(x, x) = 0$ for all x other than 0 or ∞;
(2) $F(0, x) = 1$ for all x other than 0 or ∞;
(3) $F(x, 0) = \infty$ for all x other than 0 or ∞;
(4) $F\{1, F\{1, F(1, x)\}\} = x$ for all x other than 0 or ∞.

Now, consider a function $G(x, y)$ which is derived from $1 - x/y$ by a transformation Φ which is continuous in general, which is one-to-one, and which leaves all algebraic functions algebraic. We have already shown that any such function $G(x, y)$ must be of the form

$$\frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}.$$

Let us now subject this function to a linear transformation φ which turns the transforms by Φ of 0, 1, and ∞ back into these respective numbers. The resulting function, which we shall call $H(x, y)$, satisfies conditions (1), (2), (3), and (4). Hence we have $H(x, y) = 1 - x/y$. Hence the transformation χ formed by performing first Φ and then φ leaves invariant the function $1 - x/y$. I have shown in an earlier paper* that addition and multiplication can be obtained by the iteration of the function $1 - x/y$. Hence the transformation χ leaves these functions invariant.

Now, any continuous transformation of the complex plane which leaves multiplication invariant must leave invariant the circle of the roots of unity. In a like manner, any continuous transformation of the number-plane that keeps addition invariant must keep invariant first the set of all sets each consisting of all the rational multiples of some number, then the set of all lines each consisting of the real multiples of some number, and finally must turn into a line every line in the complex plane, since every such line can be formed by adding the same number to the product of a given complex number by a variable real number. Hence our transformation χ is an affine transformation which keeps invariant the points 0, 1, and the unit circle. There is no difficulty in showing that any such transformation is either the identity or the conjugate transformation.

It follows that Φ is obtainable by applying after the identity or the conjugate operation the linear operation φ^{-1}. This is precisely the theorem which we set out to prove.

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* *Bilinear operations generating all operations rational in a domain* Ω, ANNALS OF MATHEMATICS, March, 1920.

ON THE GENERALIZATION OF CERTAIN FUNDAMENTAL FORMULAS OF THE MATHEMATICAL THEORY OF FINANCE.

BY PROFESSOR C. H. FORSYTH.

(Read before the American Mathematical Society December 29, 1920.)

1. *Introduction.* Certain of the fundamental formulas in the mathematical theory of finance lack a generality which would add much to the usefulness of such formulas. The lack of generality is due to an assumption that the periodic returns from capital profitably invested will be invested at the same rate of interest at which the principal itself is invested. For example, the most fundamental formula in the mathematical theory of finance,

$$S = P(1 + i)^n,$$

is employed to give the amount to which a principal P will accumulate at the end of n years at compound interest at rate i. But it is assumed in its derivation that every payment of interest upon the principal will be immediately invested, that all the interest upon this interest will be immediately invested, that all the interest on this third system of interest will be immediately invested, and so on, all at the original rate of interest. A much more general formula would be obtained both in this case and in many other cases if an allowance were made for the fact that the interest upon the principal may be invested at another rate of interest.

The purpose of this paper is to derive various fundamental formulas based upon an assumption that the periodic returns from an investment will be invested at another rate of interest. A few formulas will also be given which assume that even the interest upon the investment of the regular interest payments, and all further interest income, will be invested at a third rate of interest, such as that of a savings bank. Such formulas will, of course, include as special cases the formulas ordinarily employed and even the cases where the interest payments are not invested at all.

2. *The Amount of an Investment of a Single Sum. Two Rates of Interest.* If P dollars are placed at interest at rate i for n years the annual payments of interest will constitute an

annuity of iP per year. If we assume that these payments and all further interest will be immediately invested at a rate r the total accumulations or *amount* at the end of n years will evidently be

$$S = P(1 + is_{\overline{n}|}), \tag{1}$$

to be valued at rate r. If it should prove possible to invest all interest payments at the rate of interest at which the original principal was invested, that is if $r = i$, the formula (1) reduces at once to the usual form $S = P(1 + i)^n$.

If the regular interest payments are not invested at all, that is if $r = 0$, formula (1) reduces to the form $S = P(1 + ni)$, which is the formula employed in simple interest.

If the regular interest payments at rate i were made m times a year and the interest at rate r upon these interest payments were paid and compounded t times a year the formula for the amount at the end of n years would be

$$S = P(1 + js_{\overline{n}|}^{(m)}), \tag{2}$$

to be valued at rate r/t, where j is the nominal rate of interest realized upon the original principal, and $s_{\overline{n}|}^{(m)}$ is the symbol generally adopted for the conventional amount of an annuity of 1 per annum but payable m times a year, that is,

$$s_{\overline{n}|}^{(m)} = \frac{1}{m} \frac{\left(1 + \frac{r}{t}\right)^{tn} - 1}{\left(1 + \frac{r}{t}\right)^{t/m} - 1},$$

at rate r/t, or

$$s_{\overline{n}|}^{(m)} = \frac{1}{m} \frac{s_{\overline{tn}|}}{s_{\overline{t/m}|}}, \tag{A}$$

at any rate r/t. In case t/m should prove to be fractional, the value of $1/s_{\overline{t/m}|}$ could be found in a table of values of $1/s_{\overline{n}|}$ for fractional values of n or in a table of values of $i/j_{(p)}$, where

$$\frac{1}{s_{\overline{1/p}|}} = p \cdot \frac{i}{j_{(p)}}.$$

If it were found possible to invest all the interest payments at the same rate of interest and at the same frequency of conversion at which the original principal was invested, that is if $r = j$ and $t = m$, the formula (2) would reduce to the form

$$S = P(1 + j/m)^{mn},$$

which is the formula ordinarily employed in compound interest

expressed in terms of a nominal rate of interest payable several times a year.

3. *The Investment of a Single Sum. Three Rates of Interest.* It is certainly possible for the regular interest payments to be sufficiently large to enable the investor to invest them at a more favorable rate than he could realize at his savings bank, in which case he would be interested in the *amount* of his investment where three rates of interest would be involved. If the rate of interest realized on the original principal is j payable m times a year, the rate on these interest payments is r payable t times a year, and the rate on the second and all successive systems of interest payments is ρ payable θ times a year the amount at the end of n years, according to formula (2), is

$$S = P\{1 + \Sigma(j/m)(1 + rs_{\overline{x}|}{}^{(t)})\},$$

at rate ρ/θ, where the summation extends from $x = 0$ by intervals of $1/m$ to $x = n - 1/m$ inclusive. This expression reduces without difficulty to

$$(3) \qquad S = P\left\{1 + nj + \frac{jr\theta(s_{\overline{\theta/n}|} - mns_{\overline{\theta/m}|})}{mt\rho\, s_{\overline{\theta/m}|}\, s_{\overline{\theta/t}|}}\right\}.$$

It is easily verified that when $\rho = r$ and $\theta = t$ formula (3) reduces to formula (2).

4. *The Amount of an Annuity. Two Rates of Interest.* We shall consider first the amount or total accumulations of the simplest form of an annuity of 1 per annum. As the amount of an annuity is simply the sum of the amounts of all of the various payments, the amount of the annuity, which is denoted by S_n, is

$$\begin{aligned} S_n &= (1 + is_{\overline{n-1}|}) + (1 + is_{\overline{n-2}|}) + \cdots + (1 + is_{\overline{1}|}) + 1 \\ &= n + i(s_{\overline{1}|} + s_{\overline{2}|} + \cdots + s_{\overline{n-1}|}), \end{aligned}$$

at rate r, which reduces easily to

$$(4) \qquad S_n = n + (i/r)(s_{\overline{n}|} - n)$$

at rate r. If $r = i$, formula (4) evidently reduces at once to $s_{\overline{n}|}$.

If the annuity of 1 per annum were payable p times a year, the interest at rate j on these payments were payable m times a year, and the interest at rate r on these interest payments were payable and compounded t times a year the amount of the annuity would be, by (2),

$$S_n = \Sigma(1/p)(1 + js_{\overline{x}|}{}^{(m)}),$$

at rate r/t, where the summation extends from $x = 0$ by intervals of $1/p$ to $x = n - 1/p$ inclusive. This expression reduces without difficulty to

$$(5) \qquad S_n = n + \frac{jt(s_{\overline{tn}|} - nps_{\overline{t/p}|})}{pmr\, s_{\overline{t/m}|}\, s_{\overline{t/p}|}}.$$

at rate r/t.

If the annuity and all the interest payments were payable only once a year, that is if $p = m = t = 1$, formula (5) would reduce to formula (4). Similarly, if $j = r$ and $m = t$, formula (5) reduces to formula (A).

5. *The Amount of an Annuity. Three Rates of Interest.* If the annuity of 1 per annum were payable p times a year, the interest at rate j on these payments were payable m times a year, the interest at rate r on these interest payments were payable t times a year and the interest (ρ) on the second and all successive systems of interest payments were payable θ times a year the amount of the annuity, according to formula (3), would be

$$S_n = \Sigma \frac{1}{p} \left\{ 1 + xj + \frac{jr\theta(S_{\overline{\theta x}|} - mxS_{\overline{\theta/m}|})}{mtp\, s_{\overline{\theta/m}|}\, s_{\overline{\theta/t}|}} \right\},$$

at rate ρ/θ, where the summation extends from $x = 0$ by intervals of $1/p$ to $x = n - 1/p$ inclusive. This expression reduces then to

$$(6) \quad S_n = n + \frac{n\,(np-1)j}{2p}\left(1 - \frac{r\theta}{tps_{\overline{\theta/t}|}}\right) + \frac{jr\theta^2(s_{\overline{\theta n}|} - nps_{\overline{\theta/p}|})}{pmtp^2 s_{\overline{\theta/m}|} s_{\overline{\theta/t}|} s_{\overline{\theta/p}|}},$$

at rate ρ/θ.

It is easily verified that if $\rho = r$ and $\theta = t$, formula (6) reduces to formula (5). If $\rho = r = j$ and $\theta = t = m$, the formula reduces to the familiar $s_n^{(p)}$ at rate j/m.

6. *Present Values. Perpetuities.* The general formula for the present value of a single sum due at a future time where two or more rates of interest are involved is obtained by solving formula (3) for P. One must be careful, however, in problems involving several sums, to apply the formula thus obtained only when circumstances justify the application. For example, a formula so obtained should rarely be applied to compute the present value of an annuity. A little reflection will make it clear that the assumption that the same rate of interest will be realized on all interest income as on the principal may well be valid. For since the regular payment of the annuity is greater than the interest on the principal for the

same period, except in the case of a perpetuity, the one who provides the annuity needs in practice to withdraw only enough of the principal which supplemented by the interest will just provide the regular payment of the annuity. If the intervals between the payments of the annuity and the intervals of conversion of interest are the same, this procedure is equivalent to withdrawing the amount of the full payment of the annuity from the principal and then investing the interest at the same rate of interest at which the principal is invested. In any case the solution of the general problem of computing the present value of several sums due at various future times requires a careful analysis of the individual problem and can not be obtained in all cases by employing one general formula. The truth of this statement is further exemplified by the fact that it is apparently impossible to obtain such a formula which is concise and simple to apply, particularly in the case of an annuity.

The problem of finding the present value of a perpetuity whose payments are made at intervals of k years, where the interest payments are to be invested at a different rate of interest from that realized upon the principal, reduces simply to that of finding the principal whose interest payments will accumulate at the second rate to the successive payments of the perpetuity as they become due. If we denote such a principal by A and the successive payments by R, we may write

$$Ais_{\overline{k}|} = R,$$

$$A = \frac{R}{is_{\overline{k}|}}, \tag{7}$$

at rate r, which is the same formula as that ordinarily employed but is to be valued at another rate r (instead of i).

If the principal is invested at rate j payable m times a year and these interest payments and all further interest payments are invested at rate r payable t times a year the principal necessary to provide a perpetuity of R at intervals of k years is

$$A = \frac{R}{is_{\overline{k}|}^{(m)}}. \tag{8}$$

The formula corresponding to three rates of interest would be obtained by replacing $s_{\overline{k}|}^{(m)}$ by that portion of formula (3) included in the brackets minus unity. The formula would be valued, of course, at the third rate of interest.

7. *Valuation of Bonds. Two Rates of Interest.* We shall now derive formulas for valuating a bond which are based upon the assumption that the amortization factors will be invested at another rate of interest than that expected to be realized upon the principal of the bond.

We shall consider first the simplest type of a bond where the face value C is to be repaid in a single sum at the end of n years. If we denote the dividend rate offered in the bond offering by g, the rate to be realized by i, where both interests are payable but once a year, and the corresponding price by A, the amortization factor is $Cg - Ai$. If we treat the amortization factor as the annual payment into a sinking fund which is to accumulate at another rate of interest r and which is to account finally for the reduction in principal or $A - C$, we have

$$(Cg - Ai)s_{\overline{n}|} = A - C,$$

at rate r, where, in the case the bond is bought at a discount, the amortization factor and the reduction in principal are both negative. Solving for A, we find

$$A = C\,\frac{1 + gs_{\overline{n}|}}{1 + is_{\overline{n}|}},$$

at rate r. If we let $C = 1$, A becomes the price per dollar or $1 + k$, where k is the premium or discount per dollar. Solving for k, we obtain

$$k = \frac{g - i}{\dfrac{1}{s_{\overline{n}|}} + i}, \tag{9}$$

at rate r. If it should prove to be possible to invest the amortization factors at the rate planned to be realized on the whole bond, that is, if $r = i$, formula (9) reduces to the formula

$$k = a_{\overline{n}|}(g - i), \tag{B}$$

at rate i, since

$$\frac{1}{s_{\overline{n}|}} + i = \frac{1}{a_{\overline{n}|}},$$

at rate i. Formula (B) is the formula ordinarily employed to value a bond. If all the interests are payable m times a year, the formula for k is easily found to be

$$k = \frac{g - j}{\dfrac{m}{s_{\overline{mn}|}} + j}, \tag{10}$$

at rate r/m, where j is the nominal rate to be realized.

If the principal of a bond is to be repaid in installments it is easily shown that the premium or discount per dollar on the whole bond is simply the weighted arithmetic average of the premiums or discounts per dollar corresponding to the various installments treated as principals of distinct bonds where each premium or discount is weighted by the amount of the corresponding installment. For, if $C_1, C_2, \cdots, C_s$ are the values of the installments and $k_1, k_2, \cdots, k_s$ are the corresponding premiums or discounts per dollar found by one of the formulas derived above, the total price of the whole bond, say $C(1 + k)$, where k is the premium or discount per dollar on the whole bond, is given by the formula

$$C(1 + k) = C_1(1 + k_1) + C_2(1 + k_2) + \cdots + C_s(1 + k_s).$$

But $C = C_1 + C_2 + \cdots + C_s$. Therefore

$$k = \frac{C_1k_1 + C_2k_2 + \cdots + C_sk_s}{C}. \tag{11}$$

Formulas (10) and (11) can then be employed to value any bond whether the principal is to be repaid in a single sum or in installments.

Formula (10) is especially valuable because it can be applied so easily to solve the inverse problem, namely, given the price of a bond to find the rate of interest which will be realized, since on clearing of fractions j occurs only to the first degree. Those who are familiar with the difficulties encountered in employing other formulas for this purpose will appreciate the advantage to be gained in employing formula (10). This application of formula (10) is so important that the solution of the formula for j is given as follows:

$$j = \frac{g - \dfrac{km}{s_{\overline{mn}|}}}{1 + k}, \tag{12}$$

at rate r/m.

Incidentally, if the amortization factors are not invested at all, that is if $r = 0$, formula (12) reduces to the form

$$j = \frac{g - \dfrac{k}{n}}{1 + k}. \tag{13}$$

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THE SPREAD OF NEWTONIAN AND LEIBNIZIAN NOTATIONS OF THE CALCULUS.

BY PROFESSOR FLORIAN CAJORI.

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THE bitter controversy over the invention of the calculus is generally supposed to have eliminated the Leibnizian notation from Great Britain down to the time of Woodhouse, Peacock, Herschel and Babbage, and to have prevented the use of the Newtonian notation on the European continent. It is the purpose of this article to point out the extent to which this general impression is in need of revision and also to indicate the spread of these notations in America.

1. *Leibnizian Notation in Great Britain.* Several years before Newton permitted his theory of fluxions and his notation for fluxions and fluents to see the light of day in printed form, his friend, John Craig, used the notation of Leibniz, dp, dx, dy, in a book, the *Methodus Figurarum* published in 1685 in London. Craig employed the Leibnizian notation again in 1693 in another booklet, the *Tractatus Mathematicus*, as well as in articles printed in the PHILOSOPHICAL TRANSACTIONL OF LONDON for the years 1701, 1703, 1704, 1708; his article of 1703 contains the sign of integration $\int$. In the TRANSACTIONS for 1704–05, an article by John Bernoulli makes extensive use of the Leibnizian signs. In 1716 the English physician and philosophical writer, George Cheyne, brought out in London the *Philosophical Principles of Religion*, Part II, which contains a chapter from the pen of John Craig on a discussion of zero and infinity, dated September 23, 1713. In this chapter Craig uses Leibniz's symbols for differentiation and integration. But in 1718 Craig made a complete change. In that year he issued a book, *De Calculo Fluentium*, in which he switches over to the exclusive use of the Newtonian notation. Evidently this change is a result of the controversy then raging between the supporters of Newton and Leibniz.

In volume 23 of the PHILOSOPHICAL TRANSACTIONS OF LONDON, for the years 1702 and 1703, De Moivre uses dots for fluxions, but in integration he uses the Leibnizian sign $\int$. Thus, on page 1125, De Moivre writes "adeoq; $\dot{q} = \frac{3}{2}dv^2\dot{v} - \frac{3}{2}dv^2\dot{y}$, igitur $q = \frac{1}{2}dv^3 - \int \frac{3}{2}dv^2\dot{y}$. Ergo ad hoc perventum est ut

fluentum quantitatem inveniamus cujus fluxio est $\frac{3}{2}dv^2\dot{y}$." Evidently his partisanship favoring Newton did not at this time prevent his resorting to the convenience of using the Leibnizian symbol of integration.

Even John Keill, who fought so prominently and unskillfully on the side of Newton, employed a similar mixed notation. In a paper, dated November 24, 1713, and printed in the PHILOSOPHICAL TRANSACTIONS for 1714–16, he adopts the symbolism "$\int \phi\dot{x}$". A mixture of Continental and British symbols is found in the writings of E. Waring and much later in Olinthus Gregory's *Treatise of Mechanics* (3rd Ed., London, 1815) where on page 158 there is given "$\int dy \sqrt{(\dot{x}^2 + \dot{y}^2)}$." In times still more recent the Newtonian dot has been used to advantage, by the side of Leibnizian symbols, in the treatment of mechanics and the algebras of vectors.*

Returning to the eighteenth century, we find Benjamin Robins† writing the Leibnizian signs in a review of a publication of Leonhard Euler. Joseph Fenn, an Irish writer, little known, who had studied on the Continent, and was at one time professor of philosophy in the University of Nantes, issued at Dublin soon after 1768, a *History of Mathematics*. In it Fenn has occasion to use the calculus, and employs the Leibnizian notation. In a *Plan of the Instructions given in the Drawing School established by the Dublin Society* (p. LXXXIX of the above volume) he discusses the tides, and, in finding the greatest height of the tides, he uses the calculus and the notation of Leibniz. He uses them again in his *Second Volume of the Instructions given in the Drawing School established by the Dublin Society*, Dublin, 1772. He is friendly to Newton, uses the terms "fluxion" and "fluent," but never uses Newton's notation. He is perhaps the last eighteenth century writer in Great Britain who used the symbolism of Leibniz.

From our data it is evident that the Leibnizian notation was the earliest calculus notation in England which appeared in print, that in rare instances it was used by certain Englishmen, but that it vanished almost completely from British soil in the latter part of the eighteenth century.

* See, for example, P. G. Tait's *Elementary Treatise on Quaternions*, 2nd Ed., 1873, Chapter X; T. Hayashi in this BULLETIN, vol. 26, 1919, p. 74.

† Benjamin Robins, *Remarks on Mr. Euler's Treatise on Motion, Dr. Smith's Compleat System of Opticks, and Dr. Jurin's Essay upon Distinct and Indistinct Vision*, London, 1739.

2. *Newtonian Notation in Continental Europe.* That Newton's notation would be reproduced in editions of his works printed on the European continent is to be expected. Thus the fluxional symbols appear in J. Castillon's edition of Newton's mathematical works, brought out at Lausanne and Geneva in 1744, and in the appendix to the second volume of Castillon's edition of Newton's *Arithmetica Universalis*, Amsterdam, 1761. John Muller, a German by birth, who was appointed master of the Royal Military Academy at Woolwich and became "the scholastic father of all the great engineers this country (England) employed for forty years," brought out in 1736 in London a book entitled *A Mathematical Treatise: Containing a System of Conic-Sections; with the Doctrine of Fluxions and Fluents.* Muller translated his own book into French and had it published at Paris in 1760. The French as well as the English editions used the Newtonian symbols. An article from the pen of David Gregory at Oxford, after its appearance in the PHILOSOPHICAL TRANSACTIONS OF LONDON, was republished in 1697 in the ACTA ERUDITORUM of Leipzig, the British fluxional symbols being faithfully reproduced. This procedure is no more unusual than was the printing of John Bernoulli's article in the PHILOSOPHICAL TRANSACTIONS of 1704–05. The fluxional notation is given in full in the article "Fluxions" in Alexandre Savérien's *Dictionnaire Universel de Mathématique et de Physique*, Paris, 1753. We proceed to more startling disclosures.

After the middle of the eighteenth century when feelings of resentment ran high between the adherents of Leibniz and their opponents, it is extraordinary that a mathematical journal should have been published on the European continent for fifteen years which often uses the Newtonian notation, but never uses the Leibnizian. It is more surprising still, that no reference is made to this fact in any history of mathematics, though many volumes have been written which deal with various phases in the evolution of the calculus. The journal in question is a monthly, published at Amsterdam from 1754 to 1769. It appeared under the title, *Maandelykse Mathematische Liefhebbery* (Monthly Mathematical Recreations). Each issue contained about forty pages. It is mentioned in Cantor's history,* but otherwise has not attracted

* M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. IV, Leipzig, 1908, p. 70.

the attention of writers on the general history of mathematics. This neglect is due, probably, to its appearing in the Dutch language and devoting itself to the solution of elementary problems on algebra, arithmetic and geometry, and to the publication of school items of only local interest. It contains problems on maxima and minima, solved with the aid of fluxions dressed in the familiar garb of the Newtonian dots. The first twelve volumes of the journal give altogether forty-eight or more such problems; the last few volumes contain no fluxions. About seventeen different writers contributed fluxional solutions. Some writers give only their initials; none of them are men of note. It was a time when the Netherlands did not have great mathematicians. Huygens had died in 1695; Hudde in 1704; Nieuwentijt in 1718. Among the contributors using fluxions were G. Centen, J. Schoen, J. Kok, J. T. Kooyman, F. Kooyman, A. Vryer, J. Bouman, D. Bocx. The two most noted where P. Halcke, and Jakob Oostwoud. The latter was editor of the monthly during the first twelve years. Oostwoud was a teacher of mathematics in Oost-Zaandam near Amsterdam, who in 1766 became member of the Hamburg Mathematical Society.* Later he collected and published the problems solved in his journal in three separate volumes.†

3. *Calculus Notations in the United States.* In the United States the early influence was predominantly English. Wherever higher analysis received attention, it was in the form of the fluxions of Newton. Before 1766 occasional studies of conic sections and fluxions were undertaken at Yale‡ under the inspiring leadership of President Clap. Perhaps as early as 1795, and again in 1802, Jared Mansfield published at New Haven, Conn., a volume of *Essays, Mathematical and Physical,* which includes a part on "fluxionary analysis." In a footnote, on page 207, he states that the fluxions of x, y, z are

* Bierens de Haan in *Festschrift, herausgegeben von der Mathematischen Gesellschaft in Hamburg, anlässlich ihres 200 jährigen Jubelfestes 1890.* Erster Teil, Leipzig, 1890, p. 79.

† Bierens de Haan, op. cit., p. 80, where the titles of the three publications are given in full. See also de Haan's *Bouwstoffen voor de Geschiedenis der Wis-en Natuurkundige Wetenschappen in de Nederlanden,* 1878, p. 76–85, reprinted from *Verslagen en Mededeelingen der Kon. Akademie van Wetenschappen,* Afd. Natuurk., 2e Reeks, Deel VIII, IX, X en XII.

‡ *Yale College; a Sketch of its History,* by William L. Kingsley, Vol. II, pp. 497–498.

usually denoted "with a point over them, but we have here denoted these by a point somewhat removed to the right hand," thus $x^{\cdot}$. A similar variation occurs in Robert Woodhouse's *Principles of Analytical Calculations*, Cambridge (England), 1803, p. XXVII, where we read, "Again $\overset{\cdot\cdot\cdot}{\overline{xy}}$ or $(xy)^{\cdot\cdot\cdot}$ is not so convenient as $d^3(xy)$."

In 1801 Samuel Webber, then professor of mathematics and later president of Harvard College, published his *Mathematics compiled from the Best Authors*. In the second volume he touched upon fluxions and used the Newtonian dots. This notation occurred also in the TRANSACTIONS OF THE AMERICAN PHILOSOPHICAL SOCIETY, of Philadelphia, in an article by Joseph Clay who wrote in 1802 on the "Figure of the Earth." It is found also in the second volume of Charles Hutton's *Course of Mathematics*, American editions of which appeared in 1812, 1816(?), 1818, 1822, 1828 and 1831.

An American edition of S. Vince's *Principles of Fluxions* appeared in Philadelphia in 1812. That early attention was given to the study of fluxions at Harvard College is shown by the fact that in the interval 1796–1817 there were deposited in the college library twenty-one mathematical theses which indicate by their titles the use of fluxions.* These were written by members of Junior and Senior classes. The last thesis referring in the title to fluxions is for the year 1832.

At West Point, during the first few years of its existence, neither fluxions nor calculus received much attention. As late as 1816 it is stated in the West Point curriculum that fluxions were "to be taught at the option of professor and student." In 1817, Claude Crozet, trained at the Polytechnic School in Paris, became teacher of engineering. A few times, at least, he used in print the Newtonian notation, as, for instance, in the solution, written in French, of a problem which he published in the PORTICO, of Baltimore, in 1817. Robert Adrain, later professor in Columbia College and also at the University of Pennsylvania, used the English notation in his earlier writings, for example, in the third volume of the PORTICO, but in Nash's LADIES AND GENTLEMEN'S DAIRY, No. II, published in New York in 1820, he employs the dx.

A slight deviation from the Newtonian forms is found in

* Justin Winsor, *Bibliographical Contributions*, No. 32. Issued by the Library of Harvard University, Cambridge, 1888.

articles on the theory of fluxions, contributed by Elizur Wright of Tallmadge, Ohio, to the AMERICAN JOURNAL OF SCIENCE AND ARTS, for the years 1828, 1833, 1834. Like Mansfield, he does not place the dot exactly over the variable, but a little to the right, where ordinarily exponents are put. Thus, the fluxion of x is $x^{\cdot}$; the first, second and third fluxions of x^3 are, respectively, $3x^2x^{\cdot}$; $6x\,x^{\cdot 2}$, $6x^{\cdot 3}$. After 1830 the Newtonian fluxional system is rarely encountered in the United States.

In the AMERICAN JOURNAL OF SCIENCE AND ARTS, the earliest paper giving the Leibnizian symbols was prepared by Professor A. M. Fisher of Yale College. It is printed in Volume 5 (1822) and is dated August, 1818.

The earliest articles in the MEMOIRS OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES which contain the "d-istic" signs bear the date of 1818 and were from the pens of F. T. Schubert and Nathaniel Bowditch.* The latter came to know the calculus on long sea voyages during the years 1795 to 1804, when he studied Lacroix's Calculus. Bowditch began his translation of Laplace's *Méchanique Céleste* in 1814. In the MEMOIRS mentioned above, Theodore Strong of Rutgers College began in 1829 to publish a series of papers containing solutions of problems, involving the Leibnizian symbolism.

The articles contributed by Fisher, Schubert, Bowditch and Strong were isolated papers which were not accessible to the mass of students in mathematics. The publication which placed the Leibnizian calculus and notation within reach of all and which marks the time of the real beginning of their general use in the United States, was the translation from the French of Bezout's *First Principles of the Differential and Integral Calculus*, made by John Farrar of Harvard University in 1824. Between 1824 and 1831 five mathematical theses were prepared at Harvard which contain in their titles the words "Differential Calculus" or "Integral Calculus."† After 1824 French influences dominated all instruction in higher analysis at Harvard.

From what we have stated it appears that the movement toward the introduction of continental analysis began in America about ten years later than it did in England.

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* MEMOIRS OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES, Vol. 4, Part I, Cambridge, Mass., 1818, pp. 5, 47.

† Justin Winsor, op. cit.

GROUP THEORY REVIEWS
IN THE
JAHRBUCH ÜBER DIE FORTSCHRITTE DER MATHEMATIK

BY PROFESSOR G. A. MILLER.

A few erroneous statements found in a recent volume of the JAHRBUCH ÜBER DIE FORTSCHRITTE DER MATHEMATIK and relating to the theory of groups led the present writer to make a brief survey of the other volumes of this series for the purpose of determining the reliability of the reviews relating to this particular subject. As these reviews are so widely read and so frequently referred to, it seems desirable to note here a few of the instances of inaccurate or misleading statements found therein, since some of these instances may be instructive and since such a notice may tend to prevent a repetition of these particular errors. The fact that a few of the many reviewers failed to maintain the high standards of this classic series should not be surprising, and a knowledge of their shortcomings can only increase the usefulness of this series, which has had no serious competitor, since its inauguration about half a century ago, in its field of providing critical reviews of the entire current mathematical literature relating to important advances.

The erroneous statements mentioned above are found on page 164 of volume 44 (1918) and relate to the possible sets, or systems, of independent generators of a finite group whose order is a power of a prime. It is there stated that the number of the operators in such a set is equal to the index of the commutator subgroup. In the following sentence the equally incorrect statement is made that a system of independent generators of such a group can be obtained by taking one operator from each of the co-sets with respect to the commutator subgroup. According to these statements the cyclic group of order p^m, p being a prime number, would have a set of p^m independent generators, since the commutator subgroup of this group is the identity. On the contrary, it is well known that only one operator can appear in such a set of generators.

These statements are so obviously incorrect that one might be inclined to attribute them to a harmless and amusing over-

sight if they were not followed by a sentence which is not only incorrect but also practically meaningless. In that sentence, it is stated that the author in question considered the series of commutator subgroups and made assertions about their minimal order. As a matter of fact minimal orders of the *centrals* of these commutator subgroups were considered in the article under review. From what has been noted, it is evident not only that the reader of this particular review will get very little knowledge therefrom regarding the significance of the article in question, but that he is also in danger of being led to regard a theorem as true which he will find later to be false.

It may be of interest to note in this connection that each one of the published volumes of the JAHRBUCH contains reviews of articles on group theory and that more than thirty different men wrote these reviews. Most of these men published only a few such reviews. Among the early reviewers along this line E. Netto stands out most prominently for his long service and the number of articles he reviewed. In particular, Netto reviewed most of C. Jordan's articles on this subject as well as his now classic *Traité des Substitutions*, 1870. These reviews of Jordan's work were not always fair. For instance, in the review of this *Traité*, Netto said, "there are still many gaps to be filled, much diffuseness to be removed, and many proofs to be corrected."

The fact that Jordan was not pleased with this review is clear from a note published in volume 79 of CRELLE (p. 258), where Jordan refers to the first publication of Netto in support of the criticism quoted above, and says that Netto was not entirely successful in his effort. Jordan adds that he could not hope that his *Traité des Substitutions* was free from inexact statements in view of its great extent and of the novelty and difficulty of the subject treated, but that he was convinced that there were few such statements; he himself had found two which he corrected promptly and he would be greatly obliged to those who would point out others.

In a review which appeared on page 41 of the first volume of the JAHRBUCH, Netto makes an incorrect statement relating to a fundamental question, and hence it may be worth noting here. After defining the term *isomorphic group* in accord with our present definition of *simply isomorphic group*, Netto considers n rational functions $F_1, F_2, \cdots, F_n$ of the m variables x_1,

$x_2, \cdots, x_m$, which constitute a complete set of conjugates under a substitution group G on these m variables. He observes correctly that the substitutions of G transform F_1, $F_2, \cdots, F_n$ according to a transitive substitution group, but he adds incorrectly that this transitive group is isomorphic (simply isomorphic) with G. It is now well known that a necessary and sufficient condition that the transitive group on the F's be simply isomorphic with G is that the subgroup composed of all the substitutions of G which transform into itself one of these F's should be non-invariant under G and should not include any invariant subgroup of G besides the identity.

An article of unusual importance in the development of group theory is the one in which A. Capelli introduced the concept of general isomorphisms between two groups. It is therefore of interest to note here several inaccuracies which appear in the review of this article, published on page 106 of volume 10 of the JAHRBUCH. Near the beginning of this review the term subgroups is used to denote co-sets, and the definition of general isomorphisms is obscured thereby. Near the end of the review the statement that a transitive group of order p and of degree $n > p$ cannot be primitive is found. It was known even at the time of A. L. Cauchy that the order of a transitive group is always divisible by its degree and that therefore there can be no transitive group of order p and of degree $n > p$. In the same review, theorems relating to Sylow subgroups, which were well known at a much earlier date, were spoken of as if they were new.

On page 116 of volume 16 it is stated that in groups of order $n = p^\alpha \cdot q^\beta \cdot \cdots$, where $p, q, \cdots$ are prime numbers, which contain only one subgroup of each of the orders p^α, $q^\beta, \cdots$, all the substitutions are commutative. On the following page, it is said that these groups are the only ones in which every subgroup is invariant. As a matter of fact the non-abelian groups in which every subgroup is invariant were investigated much later and were named Hamiltonian groups by R. Dedekind. It is true that each of these groups contains only one Sylow subgroup for every prime which divides the order of the group, but this is not a sufficient condition for a Hamiltonian group. It is fortunate that the statement just noted was not accepted as true since otherwise the interesting later investigations relating to Hamiltonian groups would probably not have been made.

It is interesting to note that some of the most important articles were reviewed very briefly. For instance, Netto devoted only eight lines to the review of the important article by L. Sylow which appeared in volume 5 of the MATHEMATISCHE ANNALEN. A special case of a fundamental result contained in this article but which Netto failed to mention was later credited to Netto in a review written by F. Meyer and published on page 139 of volume 20 of the JAHRBUCH. The latter review is such a mixture of meaningless statements and trivialities that it would be almost amusing if it did not appear in a work which maintains, on the whole, very high standards.

In view of the fact that most of the reviewers for the JAHRBUCH have been more familiar with the work of German authors than with that of authors of other lands, it is only natural that one sometimes finds undue credit given to the former. This does not imply that these reviewers were conscious of any unfairness in giving credit. An interesting example of this apparently unconscious bias is found in the review of an article by the present writer, on page 98 of volume 27. It is there stated that the theorem that every group of order p^α, p being a prime number and $\alpha > 3$, contains an abelian subgroup of order p^3, is a consequence of theorems due to Hölder and Frobenius. As a matter of fact Hölder needed this theorem but failed to find it when he was determining the groups of order p^4, and he devoted more than 30 pages of the MATHEMATISCHE ANNALEN (vol. 43) to a discussion of properties of these groups. When the present writer communicated this theorem to Hölder, the latter replied that after receiving this information he at once went over his groups of order p^4 and verified that each of them actually contains an abelian subgroup of order p^3. He realized that the use of this theorem would have greatly simplified his discussions. The theorem might be said to be a consequence of developments due to Sylow rather than of work done by Hölder and Frobenius.

UNIVERSITY OF ILLINOIS,
December 22, 1920.

RECENT BOOKS ON VECTOR ANALYSIS.

Einführung in die Vektoranalysis, mit Anwendungen auf die mathematische Physik. By R. Gans. 4th edition. Leipzig and Berlin, Teubner, 1921. 118 pp.

Elements of Vector Algebra. By L. Silberstein. New York, Longmans, Green, and Company, 1919. 42 pp.

Vektoranalysis. By C. Runge. Vol. I. *Die Vektoranalysis des dreidimensionalen Raumes.* Leipzig, Hirzel, 1919. viii + 195 pp.

Précis de Calcul géométrique. By R. Leveugle. Preface by H. Fehr. Paris, Gauthier-Villars, 1920. lvi + 400 pp.

The first of these books is for the most part a reprint of the third edition, which has been reviewed in this BULLETIN (vol. 21 (1915), p. 360). Some small condensation of results has been accomplished by making deductions by purely vector methods. The author still clings to the rather common idea that a vector is a certain triple on a certain set of coordinate axes, so that his development is rather that of a shorthand than of a study of expressions which are not dependent upon axes.

The second book is a quite brief exposition of the bare fundamentals of vector algebra, the notation being that of Heaviside, which Silberstein has used before. It is designed primarily to satisfy the needs of those studying geometric optics. However the author goes so far as to mention a linear vector operator and the notion of dyad. The exposition is very simple and could be followed by high school students who had had trigonometry.

The third of the books mentioned is an elementary exposition of vectors from the standpoint of Grassmann. The author uses Gibbs' notation, abandoning the usual German notation of parentheses and brackets. This is a step in the right direction, though matters would be still further improved by using the Hamiltonian notation complete. The development of the *tensor* (dyadic)* is after Gibbs. The treatment is entirely

* *Tensor* as here used means dyadic, though generally it means *symmetric dyadic, self-transverse matrix.*

for ordinary space, consideration of four-dimensional and higher space, useful for relativity, being deferred to the succeeding volume.

The first chapter is on vectors and plane magnitudes or quantities, and defines the various kinds of products. The second chapter is on differentiation and integration. The third is on tensors or dyadics. Simple applications are introduced where needed, the development reaching the usual elementary theorems on divergence, curl, Stokes' theorem and Green's theorem. The author uses the vector directly and only occasionally reverts to vectors as triples. There is not much attempt to elaborate formulas.

The fourth book is all that the title implies, and as a manual for working mathematicians and students is quite complete. It was written during the captivity of the author. There is a carefully worked out plan of exposition, from general notions on complex numbers to applications to electricity. In a preface of fifteen pages the general plan is explained with reasons for the eclectic notation. The sixteen chapters are analyzed, article by article, in a table of contents, making the work very accessible to the reader who wishes to consult the book. This table occupies twenty-six pages. An introduction of four pages examines the idea of quantity, and gives certain historical references of use. We shall give some idea of the scope of the work.

The first chapter is fundamental. In it the author starts with the notion of *qualitative* number as a given foundation, defines units, and takes as primary notion that of number considered to be the sum of numerical multiples of the units. The totality of numbers thus formed for a given set of units constitute the *domain* of the units. Domains are identical when every number (hypernumber) in either is also in the other. The *degree* of the domain is the number of linearly independent units necessary to express every number of the domain. Units are not unique, in the sense that any system of numbers of the domain which are linearly independent and in number as many as the degree of the domain, will equally define the domain. The divergence of Hamiltonian developments from Grassmannian developments comes with the idea of the product of two hypernumbers. In the former a product is always in the domain of the total set of units; in the latter a product is a hypernumber of a different grade and not in the

domain. This very great distinction is usually omitted, overlooked, or not regarded as existing, in many treatments of vectors. It is, however, vital.

In the second chapter are defined vector, parallelogram (bivector) and *spath* (trivector). Addition of points and the barycentric calculus are defined, and the equations that result are deduced. The bipoint or glissant vector and the tripoint or glissant bivector are defined. The third chapter deals with combinatorial or alternating multiplication, progressive and regressive, exterior and interior. The fourth chapter applies these notions to geometry, kinematics, and statics. The notions are essentially those of Grassmann.

Chapters five and six consider the product of vectors and the scalar and vector parts of the products. These notions are essentially quaternionic. The fundamental formulas are derived from the postulated laws of multiplication of the unit vectors i, j, k. Chapter seven deals with the differentiation of vectors.

Chapter eight applies all the foregoing developments to geometry, kinematics, twisted curves, and curved surfaces. In each case the fundamental theorems are proved and the necessary useful formulas are deduced.

Chapter nine develops in some detail the dyadic or linear vector operator. The notation of Gibbs is introduced in part. The invariants, scalar and vector, are found, and important special cases are studied. An application is made to quadrics.

Chapters ten and eleven consider the very important *nabla* of Hamilton, and the various resulting differential expressions. In chapter twelve multiple integrals with the theorems of Stokes and Green, and a study of vector fields, are to be found. These three chapters lay the foundations for mathematical physics. In the remaining chapters the theory of the potential, movement of a solid, elasticity, and electromagnetism, are treated.

The book is amply supplied with examples and exercises. The typography is up to the usual excellence of the publications of the firm, and few errors occur. As a satisfactory beginning book for the various forms of vector calculus it is to be highly recommended.

JAMES BYRNIE SHAW.

SHORTER NOTICES.

Oeuvres de G. H. Halphen. Publiées par les soins de C. Jordan, H. Poincaré, E. Picard, avec la collaboration de E. Vessiot. Vol. I, 1916, xliv + 570 pp. Vol. II, 1918, vii + 560 pp. Paris, Gauthier-Villars.

Picard in his *Notice sur la vie et les travaux de Georges-Henri Halphen* and Poincaré after him in his *Notice sur Halphen* urged that one could distinguish among the mathematicians of thirty years ago two well-marked and opposing tendencies of thought separating mathematical labors into two distinct categories and the mathematicians themselves, perhaps less definitely, into two classes. Those of the one class are preoccupied principally with enlarging the field of known notions; the others prefer to devote their energies primarily to penetrating more deeply into notions which already have been analyzed and elaborated. This distinction of classes is just as valid today as it was when Picard and Poincaré insisted upon it in 1890; and it is likely to remain so as long as mathematics develops along the lines already marked out.

Those who are most concerned with extending the frontiers of science often find it necessary to leave their new ideas without much elaboration and to proceed to a general account of the lay of the field, so to speak, in order to obtain at once a comprehensive view. In such investigations many questions will be raised and not answered, it is very difficult to stand always clear of errors in detail, lines of investigation which deserve to be followed up must at most be only indicated, and many promising thoughts must be dismissed altogether.

The mathematicians of the second class, of whom Halphen is one of the greater, are more intimately concerned with a desire to give to their work a character of absolute perfection. An error with respect to even the smallest detail becomes a matter of acute pain. Whatever these mathematicians touch they wish to achieve to the point of leaving unanswered no question which their investigation raises. They seek nothing less than to put their thought into a form of absolute perfection and beauty.

These two directions of mathematical thought are observed also in the different branches of the science. One can probably

still say (as Picard said in 1890) that in a general way the tendency to comprehensive investigation and broad characterization is found more often in matters relating to the theory of functions, and the tendency to deep penetration in the works on modern algebra and analytic geometry.

The work of Halphen is marked principally by the tendency to deep penetration. He devoted himself for the most part to algebra and geometry, fields apparently best suited to such a temperament. He analyzed the difficult problems attacked to the point of complete and definitive results. Everywhere his work is marked by the effort to leave not a single thing unachieved.

After the researches of de Jonquières and Chasles there was a lively investigation of algebraic systems of conics depending on a single parameter. Chasles had found, by a sort of induction, a general law giving the number of conics satisfying a given condition. This number was composed of a sum of two terms, each of them being a product of two factors one of which depended on the system and the other on the condition. Halphen, simultaneously with several other mathematicians, sought a proof of the law of Chasles; and he believed that he had found one. Later he perceived an error in his argument and consequently took up again a study of the question. After a long investigation his labors were rewarded through the discovery of a complete solution of the problem by means of a method of great originality. The result is distinguished by a characteristic deep penetration.

Halphen's masterpiece, according to Poincaré, is his memoir on twisted algebraic curves, crowned in 1881 by the Berlin Academy. This theory is contrasted in a remarkable manner with that of plane curves. For the latter a single number, the degree, suffices for a complete classification. Twisted algebraic curves do not possess a like property. To give a single number is not enough to classify them. Moreover, one cannot find a system of integers enjoying with respect to twisted curves a rôle analogous to that of the degree for plane curves. Halphen obtains a satisfactory solution of the problem; but, since the solution is not susceptible of an analytical expression, its character is likely to escape the superficial reader. The results bring to light certain curious and unnoticed properties which are brought out clearly by Poincaré (vol. I, p. xxxi).

Among the other larger subjects investigated by Halphen

may be mentioned the following: singular points of algebraic curves, differential equations and invariants, elliptic functions, theory of numbers, and theory of series.

For a brief sketch of the life of Halphen, with some remarks on the character of his work, the reader may consult the *Notice* by Picard (vol. I, pp. vii–xvi). In the *Notice* by Poincaré (vol. I, pp. xvii–xliii) we have an excellent systematic analysis of his mathematical contributions. Besides this there is the *Notice* by Halphen himself (vol. I, pp. 1–47) in which his own contributions were analyzed on the occasion of his candidacy before the Paris Academy of Sciences in 1885, about four years before his death. These excellent brief accounts of his work relieve the reviewer of the duty of making an analysis of the separate memoirs. The entire works (with the exception, apparently, of the *Traité des Fonctions Elliptiques*) are to be included in four volumes, of which the third is announced as in press and the fourth in preparation.

R. D. CARMICHAEL.

A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse. By Florian Cajori. Chicago, The Open Court Publishing Company, 1919, pp. viii + 299.

This work appears as number five in the Open Court Series of Classics of Science and Philosophy, a series which should meet with all the encouragement and support that American scholars can give in these times, when the question of the publication of such works is so critical. That such encouragement and support is justified may be seen from an examination of this latest production of Professor Cajori's pen, for he has here given to scholars one of the best of his various studies in the history of mathematics.

The work consists of twelve chapters under substantially the following titles: I. Newton; II. Printed books and articles on fluxions before 1734; III. Berkeley's *Analyst;* IV. Jurin's controversy with Robins and Pemberton; V. Textbooks immediately following Berkeley; VI. Maclaurin's *Fluxions* (1742); VII. Textbooks of the middle of the eighteenth century; VIII. Robert Heath and the controversy in his time; IX. Abortive attempts at arithmetization; X. Later works on fluxions; XI. Criticisms under the influence of

French writers; XII. Merits and defects of the early fluxional conceptions. The work has a good index which students will find helpful in the matter of ready reference.

The two features of the work that will appeal to the student as of paramount importance are the selection and arrangement of original material and the translations which accompany the Latin texts. For good or ill, the ability of American scholars to read Latin easily has departed; and although the original Latin forms are desirable for reference, either a translation or an explanation of some kind has now come to be a desideratum, if not a necessity. Professor Cajori has placed students under great obligation for searching out with care the passages in Wallis, Newton, and others who played leading parts in the perfecting of the calculus, and in presenting these passages in a manner that renders them accessible to all who care to consult them. Here the reader will find the early symbolism, the definitions, and the methods of approach that are necessary to an understanding of the numerous controversies that agitated mathematical England, with respect to the calculus, for more than a century.

In the first chapter Professor Cajori has given numerous excerpts from such works of Newton as appeared before 1734, the date of Bishop Berkeley's attack upon the whole theory of fluxions. These are chiefly from the *Principia* (1687), from Newton's *Quadrature of Curves,* which Wallis first published in his Algebra (1693), from the *Commercium Epistolicum* (1717) and from certain letters and manuscripts. These extracts show that Newton made use of infinitesimals as early as 1665, that he used the dot notation in the same year, that he first used the word "fluxion" (at least in print) in 1687, that he used the idea of limits in 1687, and that he finally placed his theory on a thoroughly logical basis in the 1704 edition of his *Quadrature of Curves.* The summary is a particularly lucid one of the steps taken by Newton in his development of the calculus.

The literature upon the subject which appeared in Great Britain between 1686, when John Craig published his *Methodus Figurarum . . . quadraturas determinandi,* and 1734, when Berkeley's *Analyst* appeared, is next considered. Professor Cajori shows that the influence of Leibniz was much greater in England than is generally suspected, and shows that the clashing of the two different methods of attack, the New-

tonian and the Leibnizian, gave to Berkeley an excellent opportunity for "the most spectacular event of the century in the history of British mathematics." Mathematicians are generally aware of the fact that this attack was made, but the details are rarely known. Berkeley's *Analyst* is not a common work, although it is occasionally offered by dealers in the classics of the eighteenth century. For this reason the careful summary here given will be welcome to those who care for the genesis and early status of great movements in the field of their favorite science. The extracts from Berkeley, for example, have been carefully copied, although the author has taken certain justifiable liberties in the matter of punctuation, spelling, and capitalization.

A further service rendered by Professor Cajori is seen in his chapters on the books on the fluxional calculus that appeared after Berkeley's attack; many of them, no doubt, being the natural result of such criticism, the authors appearing as champions for a cause that seemed to them worthy of strong defence. For the interesting details of this defence, and for the subsequent fortunes of the fluxional type of analysis, however, the reader should consult the work itself.

The mathematical world has often been in debt to Professor Cajori for his detailed studies, but never more so than in this case.

DAVID EUGENE SMITH.

Cours de Cinématique théorique. By H. Lacaze. Paris, Gauthier-Villars, 1920. 138 pp.

This text is for the use of students of the lycée and the government schools. The first part of the book covers 56 pages, the complementary part 82 pages. In the first part are five chapters entitled, respectively, vectors, kinematics of a point, movement of a solid and distribution of velocities, composition of accelerations, and displacement of a vector in a plane. The complementary part has five corresponding chapters. Vectors are represented throughout as triples, no real vector notation entering the book. The first part is elementary, the complementary part is more advanced. Explanations are brief but sufficient, and a few exercises are introduced which help out the book for student use. It is clear, well-printed, and ample for a beginning course.

JAMES BYRNIE SHAW.

Some famous problems of the theory of numbers and in particular Waring's problem. An inaugural lecture delivered before the University of Oxford by G. H. Hardy. Oxford, Clarendon Press, 1920. 8vo. 34 pp.

The particular problems with which this lecture is concerned belong to the additive theory of numbers. The general problem of the latter is stated by Hardy as follows: "Suppose that n is any positive integer, and $\alpha_1, \alpha_2, \alpha_3, \cdots$ positive integers of some special kind, squares, for example, or cubes, or perfect kth powers, or primes. We consider all possible expressions of n in the form $n = \alpha_1 + \alpha_2 + \cdots + \alpha_s$, where s may be fixed or unrestricted, the α's may or may not be necessarily distinct, and order may or may not be relevant, according to the particular problem on which we are engaged. We denote by $r(n)$ the number of representations which satisfy the conditions of the problem. Then *what can we say about* $r(n)$? Can we find an exact formula for $r(n)$, or an approximate formula valid for large values of n? In particular, is $r(n)$ *always positive*? Is it always possible, that is to say, to find at least *one* representation of n of the type required? Or, if this is not so, is it at any rate always possible when n is sufficiently large?"

The number $p(n)$ of unrestricted partitions of n into positive integral summands has been studied by many authors; the principal result of the investigation of this function by Hardy and Ramanujan has been the discovery of an approximate formula for $p(n)$ which enables them to approximate to $p(n)$ with an accuracy which is almost uncanny. Of $p(200)$, for example, the value 3,972,999,029,388 is obtained with an (additive) error of .004 by employing eight terms of their series; and the result has been verified by MacMahon, without the use of their formula, by a direct computation which occupied over a month.

The principal object of the lecture is a discussion of the problem of Waring of determining the number of representations of an integer n as a sum of s positive kth powers of integers and particularly of the (more usual) restricted form of this problem in which one seeks to show that for fixed k there exists a finite s_k independent of n such that every integer n has at least one representation as a sum of s_k non-negative kth powers. In connection with this problem there are two functions of fundamental importance, whose existence has

been proved in recent years; they may be defined as follows: The number $g(k)$ is defined to be the least number for which it is true that every positive integer is the sum of $g(k)$ non-negative kth powers of integers; the number $G(k)$ is defined to be the least number for which it is true that every positive integer from a certain point onwards is the sum of $G(k)$ non-negative kth powers of integers. The existence of either of the functions $G(k)$ and $g(k)$ obviously implies that of the other. The existence of $g(k)$ was first proved by Hilbert in 1909, after an interval of 139* years from the time of its enunciation by Waring,† who gave the theorem without proof.

For a long time it has been known that $g(2) = G(2) = 4$. In 1859 Liouville proved that $g(4)$ exists and does not exceed 53; it was shown by Wieferich in 1909 that $g(4) \leqq 37$, the most that is known at present. The number $79 = 4 \cdot 2^4 + 15 \cdot 1^4$ needs 19 biquadrates, and no number is known which needs more. There is still therefore a wide margin of uncertainty as to the actual value of $g(4)$. The existence of $g(3)$ was first established in 1895 when Maillet proved that $g(3) \leqq 17$; Wieferich proved in 1909 that $g(3) \leqq 9$. As 23 and 239 require exactly 9 cubes, the value of $g(3)$ is exactly 9. [Hardy remarks that it is "no doubt true" that 23 and 239 are the only integers requiring 9 cubes for their expression.] In 1909 Landau proved the "singularly beautiful theorem" that the number of integers requiring 9 cubes each for its expression is finite. It was in view of this fact that the number $G(k)$ was introduced. It is known that $4 \leqq G(3) \leqq 8$; and Hardy is

* Corrected from Hardy's "127" on page 17, in accordance with the information indicated in the next footnote.

† At the time when Hardy wrote his address he was under the impression that Waring first stated his theorem [that every positive integer is a sum of at most 4 positive squares, 9 positive cubes, 19 positive biquadrates, and so on] in the third edition (1782) of his *Meditationes Algebraicae* [pp. 349–350], but in a letter of Jan. 4, 1921, he writes me that a correspondent has called his attention to its appearance in an earlier edition. On examining the three editions I fail to find it in the first (1762), but find it in the second (1770) [pp. 204–205], and in the third (1782), as indicated. These references are given also in Dickson's *History of the Theory of Numbers*, vol. II, pp. xviii and 717 in connection with his elaborate history of Waring's problem. [Am I right in supposing (as I have done in numbering the editions above) that Waring, when he came to publish the third edition, treated the first part (pp. 1–65) of his *Miscellanea Analytica* (1762) as the first edition of his *Meditationes Algebraicae*, its material being reproduced in the editions of 1770 and 1782; or was there another edition of the latter between those of 1770 and 1782? The view which I have taken agrees with a statement given in an old manuscript note on the fly-leaf of a copy of the 1770 edition of the *Meditationes Algebraicae* in the library of the University of Illinois.]

disposed to conjecture that $G(3)$ has the value 4 or 5, and he seems to lean towards the former (p. 23). A simple argument shows that $G(4) \geqq 16$ and Hardy and Littlewood have proved that $G(4) \leqq 21$, so that $G(4)$ lies in a comparatively small known range.

On pages 27–34 we have one of the most fascinating accounts in our literature of the fundamental idea which has guided a mathematical investigation, the account of that which directed Hardy and Littlewood in their investigation of the properties of $G(k)$. The method seems to be very powerful. It has brought them for the first time into relation with the series on which the solution in the last resort depends; it gives numerical results which, as soon as k exceeds 3, are far in advance of any known before; and it gives a definite upper bound to $G(k)$, namely,

$$G(k) \leqq (k-2)2^{k-1} + 5.$$

[On the other side it is known that $G(k) \geqq k + 1$ and that $G(2^\alpha) \geqq 2^{\alpha+2}$ if α is a positive integer.] Hardy adds: "It is beyond question that our numbers are still very much too large; and there is no sort of finality about our researches, for which the best that we can claim is that they embody a method which opens the door for more."

Concerning Goldbach's assertion that every even number is the sum of two primes we have the following (p. 34): "Our method is applicable in principle to this problem also. We cannot solve the problem, but we can open the first serious attack upon it, and bring it into relation with the established prime number theory. The most which we can accomplish at present is as follows. We have to assume the truth of the notorious Riemann hypothesis concerning the zeros of the zeta function, and indeed in a generalized and extended form. If we do this we can prove, not Goldbach's theorem indeed, but the next best theorem of the kind, viz. that *every odd number*, at any rate from a certain point onwards, *is the sum of three odd primes*. It is an imperfect and provisional result, but it is the first serious contribution to the solution of the problem."

It is with genuine regret that the reviewer has to point out one or two historical errors in an address which is otherwise so charming. (We have already mentioned one of these.) On page 18 he refers to Fermat's "notorious assertion concerning Mersenne's numbers"; a letter to the reviewer indicates that

this error probably arose through referring to Fermat a statement which was in fact made by Mersenne (and stated by W. W. Rouse Ball to be "probably due to Fermat").*

From page 18 of Hardy's lecture I quote as follows: "No very laborious computations would be necessary to lead Waring to a highly plausible speculation, which is all I take his contribution to the theory to be; and in the theory of numbers it is singularly easy to speculate, though often terribly difficult to prove; and it is only proof that counts." It is hard to see in what sense the author can say that "it is only proof that counts" when he has before him a conjecture like that of Waring which has certainly influenced for good the development of a very fascinating chapter in the modern theory of numbers. Probably the same feeling that induced this statement led to Hardy's calling by the name "theorem of Lagrange" the theorem that every integer is a sum of four non-negative squares, whereas Fermat had stated that he had a proof of the theorem (both Fermat and Bachet ascribing the theorem to Diophantus) and Euler had made repeated efforts for forty years to prove it before Lagrange through the aid of Euler's work succeeded in giving the first proof in 1772. [See Dickson's *History*, vol. II, pp. ix, x, 275–303.] It appears to me to be unfortunate to have this theorem called by the name of Lagrange; it certainly represents one extreme of judgment concerning the question of attaching names of mathematicians to specific theorems.

The opposite extreme of the same thing recently came to my attention in another connection; curiously enough, it is again a case of a "theorem of Lagrange." The theorem that the order of a subgroup is a factor of the order of the group containing it has been called the "theorem of Lagrange" by at least two authors of high repute [see Pascal's *Repertorium* (in German), vol. I, 2d edition, 1910, p. 194, and Miller, Blichfeldt and Dickson's *Finite Groups*, 1916, p. 23 (in the part written by G. A. Miller)]. Now the facts seem to be that

* If any one of the many mathematical propositions stated by Fermat is incorrect, with perhaps a single exception, I am unaware of it. The case of exception is that concerning the prime character of the so-called Fermat numbers $2^k + 1$ where $k = 2^n$; and this incorrect statement he first made several times as a conjecture and finally (after the lapse of several years) implied that he knew a proof of it. This would seem therefore to be a lapse of memory rather than an error in reasoning. All his other theorems have been proved with the one famous exception. See Dickson's *History*, vol. I, p. 375, and vol. II, p. xviii.

Lagrange knew the theorem only for the case of the subgroups of the symmetric group and that even for this case he had no satisfactory proof. Abbati (in 1803) completed the proof for subgroups of the symmetric group and also proved the theorem for cyclic subgroups of any group; but it was apparently more than seventy-five years after the publication of Lagrange's memoir (in 1770–1771) before the completed theorem became current (though it had appeared earlier in a paper by Galois in 1832). In this case we have attributed to Lagrange a theorem which he probably never knew or conjectured, on the ground (it would seem) that he knew a certain special case of it. In Hardy's paper we have a theorem referred to Lagrange apparently on the ground that he first published a proof of it though it had been in the literature long before. Somewhere between these two extremes lies the golden mean of proper practice in attaching the names of mathematicians to specific theorems; and this mean, in the opinion of the reviewer, is rather far removed from each of the extremes indicated.

R. D. CARMICHAEL.

Statics, including Hydrostatics and the Elements of the Theory of Elasticity. By Horace Lamb. Cambridge, University Press, 1916. xii + 341 pp.

Mathematics as ordinarily taught in our colleges and mathematics as used in this work-a-day world are birds of entirely different feather, and they do not flock together. This may perhaps be illustrated by a simple problem (No. 20, p. 178) from Lamb's *Statics*:

"Water is poured into a vessel of any shape. Prove that at the instant when the center of gravity of the vessel and the contained water is lowest it is at the level of the water surface."

Let us imagine a well trained sophomore attacking this problem. It is clearly a minimum problem involving integration. We measure h vertically upward from the bottom of the inside of the container, take the density as unity (or shall we keep it as ρ?), and let $A(h)$ be the area of the cross-section of the vessel. Then the center of gravity of the water is at a height

$$h_1 = \int_0^h \rho h A dh \div \int^h \rho A dh.$$

Let the mass of the vessel be denoted by M, and let its center

of gravity be at the altitude h_0. Then the center of gravity of the whole is at the height

$$H(h) = \frac{Mh_0 + \int_0^h \rho hAdh}{M + \int_0^h \rho Adh};$$

and it is to be proved that

$$\frac{dH}{dh} = 0 \quad \text{and} \quad \frac{d^2H}{dh^2} > 0 \quad \text{when} \quad h = H.$$

I have assumed perhaps naturally that the student knows enough or too little not to use double or triple integrals, but knows too much not to use any. I have also assumed that he knows enough to write $h = H$, which I doubt.

Having thus stated his problem, perhaps he can solve it and perhaps he cannot, but probably he will get at least 50% for his statement, even if he has not proved anything except that his thorough course in calculus has developed a serious set of inhibitions whereby he is prevented from using his common sense on a simple mathematical problem.

Lamb's Statics is full of such disappointments for the student of mathematics. Indeed it may be feared that this is malice aforethought on the part of the author. (See ¶ 7 of his Preface). Nor would I seem to criticize mathematical instruction of others without being more precisely critical of my own. For twenty years I have taught or tried to teach not only mathematics of the canonical sort to all grades of students but mechanics and physics to Freshmen, to Juniors, to Seniors, and to Graduates. It is my own experience that despite my best endeavors my students will solve successfully with complicated mathematical machinery problems that I am confident they do not understand and will fail lamentably in the solution of simple common-sense problems where the canonical machinery is in the way. Perhaps we should teach less of the machinery, reduce things less to rule and formal procedure, and above all dwell longer on the simple fundamentals of our subjects of instruction. Mechanics is a particularly difficult topic for teacher and for taught. In our large engineering schools we are apt to take three shots at mechanics: once in the course in physics, once at some point of the course in mathematics, and once in applied mechanics. The material covered overlaps a great deal, but the points of view are often divergent. Might it not be that if we all got together, physicist, mathematician, and engineer, pooling our

combined time, and cooperating at each phase of the work, we should accomplish a far better result?

Lamb's Statics keeps to the middle road of presenting the subject as a branch of natural science, largely deductive because of the paucity and simplicity of the fundamental laws. It is a work on physics rather than on engineering or mathematics; it should afford a fine introduction whether to applied mechanics of the more technical sort, to the theory of structures, to more advanced physical theories, or to analytic statics.

To show the breadth of treatment the titles of the chapters may be quoted: Theory of Vectors, Statics of a Particle, Plane Kinematics of a Rigid Body, Plane Statics, Graphical Statics, Theory of Frames, Work and Energy, Analytical Statics, Theory of Mass-Systems (centers of mass and moments of inertia), Flexible Chains, Laws of Fluid Pressure, Equilibrium of Floating Bodies, General Conditions of Equilibrium of a Fluid, Equilibrium of Gaseous Fluids, Capillarity, Strains and Stresses, Extension of Bars, Flexure and Torsion of Bars, Stresses in Cylindrical and Spherical Shells. This is a considerable program; it is well and consistently carried through —as should be expected by all who have known his other writings and particularly his companion volume on *Dynamics*, noteworthy for the same Greek characteristic σωφροσύνη.

These books, *Statics* and *Dynamics*, are not written for the writing; they are products of teaching, for they are based on lectures delivered at the University of Manchester. If the matter were taken slowly enough, satisfactory results would attend their use in our American institutions, provided our teachers had an all round interest in the elements of mathematics, of physics, and of engineering, and a fine contempt for superficial ground-covering in any of the three.

E. B. WILSON.

Principes usuels de Nomographie avec applications à divers problèmes concernant l'artillerie et l'aviation. Par Lieutenant-Colonel Maurice d'Ocagne. Paris, Gauthier-Villars, 1920. 67 pages.

This pamphlet, as the title indicates, is a short exposition of nomography in which the illustrations are taken from artillery and aviation. Nomography is the general theory of the graphical representation of equations of any number of vari-

ables: the end of such a representation is to replace all kinds of numerical computation by readings made on the graph. If an equation contains three variables x, y, z, say, then by giving a value to z and using x and y as rectangular coordinates a curve can be constructed. If these curves are constructed for various values of z then a set of values x, y, z satisfying the equation will be found by taking the coordinates x, y of any point together with the value of z which corresponds to the curve passing through that point. If the z-curves are constructed reasonably close together it is possible to interpolate. The corresponding values are then determined as the intersection of three curves. If then instead of point coordinates, tangential coordinates are used corresponding values will correspond to points on a line. D'Ocagne has introduced a particular kind of tangential coordinates which makes the constructions very simple. This is the alignment chart and has many advantages over the construction first mentioned. The drawing of curves is replaced by the construction of scales and, for the case of four variables, the whole drawing can be made on a single sheet of paper. The first half of the pamphlet deals with the theory of the construction of these charts.

The second part applies the theory to the actual making of charts for formulas used in artillery and aeronautics. The charts are arranged according to the simplicity of the construction. Among the charts we find the following: the coefficient of adjustment, $K = D/A$; the angle of sight, $\tan z = Z/D$; the initial angle of fire,

$$(1 + \cos 2\varphi) \tan \epsilon = \sin 2\varphi - \sin 2\alpha;$$

the limiting velocity for arming fuses, $\frac{1}{4}K\pi D^2 = p/V^2$; the interior pressure in the autofrettage,

$$\omega + 1492 = \frac{\lambda_1}{\rho^n} + \frac{\lambda_2}{\rho^2}.$$

These give an idea of the kind of formulas handled. The charts are well constructed and the explanations are very clear. On the whole this little pamphlet makes delightful reading and, now that graphical charts are so much used, the teacher of mathematics should no longer be ignorant of the subject.

C. L. E. MOORE.

Mathematische Streifzüge durch die Geschichte der Astronomie. By P. Kirchberger. Mathematische-Physikalische Bibliothek, Bd. 40. Leipzig, Teubner, 1921. iv + 54 pp.

Some of the difficulties of publication and sale in connection with German works are seen in the case of this worthy addition to the Teubner list,—one of the small monographs of the type that was intended to and did popularize science before the war, and that sold in those days for a mark or less, often in a cloth binding. This particular number, like others at the present time, appears in paper binding, with a nominal price of 2 Marks in Leipzig, but with the announcement that there is also a Teuerungszuschlag of 120% for home purposes, with a Bewilligung stamp saying that the value is 3 Marks 30 Pf., and with an accompanying statement that the price of the book to an American buyer is 10 Marks,—all of which is an interesting problem for the economist.

As to the work itself, it maintains the high intellectual standard usually found in German booklets of this class. It is written by one who holds professorial rank in the Leibniz-Oberrealschule at Charlottenburg, and it shows that scholarly ideals still prevail in spite of post-bellum difficulties.

Dr. Kirchberger has divided his work into six chapters, in each of which he has considered the essential features which mathematics has contributed to the progress of astronomy. The general topics are as follows.

I. Early history to the time of Hipparchus, including (a) the first steps in the making of a system of chronology, together with early traces of the idea of a continued fraction; (b) astronomical geography, as in the geodetic work of Eratosthenes; (c) the Greek contributions to measurements relating to the sun and the moon, as in the finding of the supposed size of and distance to the sun by Aristarchus.

II. The planetary theories of Ptolemy, Copernicus, and Tycho Brahe, including Ptolemy's theory of epicycles and the effect of improved methods of observation and computation upon the work of the sixteenth century.

III. Kepler, with a résumé of his leading mathematical contributions to astronomy, and with attention to the fact that he looked upon the latter science as a branch of physics rather than a part of geometry.

IV. Newton, with especial reference to his theory of gravity.

V. From Halley to Bessel, including particularly the de-

velopment of the modern ideas of parallax and aberration and with special attention to the contributions of Lacaille, Lalande, and Römer.

VI. Modern problems.

The work is, of course, merely the briefest kind of survey. Only forty-one names are mentioned, and under the prominent names there could not be, under the limitations imposed, more than a mere indication of the work accomplished by each. Nevertheless, for general students of mathematics and of astronomy, the book will furnish some interesting reading and some valuable information. There is a helpful bibliography, mostly of German sources, but including several translations from the English and French.

DAVID EUGENE SMITH.

Natural Tangents. By Emma Gifford. Manchester, England, 1920. iv + 90 pp.

It is now seven years since Mrs. Gifford issued her table of natural sines to every second of arc and to eight places of decimals. This appeared in the spring of 1914, before the war began. The demands upon the time and strength of everyone in England were so great during the years that followed that Mrs. Gifford's projected table of natural tangents was greatly delayed, and the present volume is only the beginning of a work that will be as complete as was the former one. The table extends only to 15°, but the progress will now be more rapid and we may hope soon to see the work completed.

Mrs. Gifford has taken the natural tangents for every 10″ from the *Opus Palatinum* of Rheticus, published in 1596. She has then found the tangents to 1″ through interpolation by the aid of a calculating machine. The computation was in each case carried to ten places for the purpose of establishing the eighth place. The arrangement is that of the Chambers logarithmic tables, which is more convenient than the semiquadrantal plan commonly found in this country, although it makes the book more bulky.

Mrs. Gifford is so well known as a careful computer that the work will be welcome to all who have need for an eight-place table of natural tangents. Such a table is suggestive of the diminishing relative importance of the logarithm with the rapid improvement in mechanical calculation.

DAVID EUGENE SMITH.

Oeuvres Complètes de Cristiaan Huygens, publiées par la Société Hollandaise des Sciences. Tome XIV. *Calcul des Probabilités. Travaux de Mathématiques Pures, 1655–1666.* La Haye, Martinus Nijhoff, 1920. 524 pp.

Approximately the first one-third (pp. 1–179) of this attractive volume is devoted to contributions to the theory of probability (1656–1657), with appendices most of which are of later date. The next division (pp. 181–407) presents contributions to various subjects (1655–1659) including the theory of numbers, rectification of the parabola, quadrature of conoids, volumes of solids of revolution, centers of gravity, properties of the cycloid, and theory of evolutes. The third division (pp. 409–427) gives contributions to the commentaries of von Schooten on the *Geometria* of Descartes, editions of 1649 and 1659. The fourth division (pp. 429–524) like the second relates to a wide range of subjects. This division (1661–1666) presents contributions to the theory of logarithms including the logarithmic curve with application to the determination of altitude by means of the barometer, further work on solids of revolution, rules for finding the tangent to an algebraic curve, and the theory of cubics. Each division is preceded by a valuable historical introduction.

Although the first steps in the theory of probability were taken by Pascal and Fermat in their correspondence as early as 1654 about questions proposed by the Chevalier de Méré, the work of Huygens constitutes the first treatise on the subject. He starts from the fundamental assumption that there exists for any equitable game a determinate numerical value for the probability that a player will win or lose. He develops a few elementary propositions of which he makes frequent application. Proposition III asserts that if a player has "p chances of gaining a and q chances of gaining b, his expectation is $(pa + qb)/(p + q)$." In present day usage, we should probably avoid the use of the word "chances" in the sense in which it is here used. The thought could be expressed by saying that $(pa + qb)/(p + q)$ is the expectation of a trial when in a total of $p + q$ equally likely trials, there are p ways of gaining an amount a and q ways of gaining b. This proposition affects much that follows it.

The entire treatment of probability contains fourteen propositions, five problems, and nine appendices. The first two propositions are special cases of proposition III just

stated. The propositions IV to IX are concerned with the problem of points with two or three players. The propositions X to XIV deal with the throwing of dice. The nine appendices treat various applications many of which came up in correspondence. The last one, which carries the date 1688, is the problem of the chances of three players at piquet. This problem was reduced by Huygens to the sum of an infinite series. This appendix was written much later than the treatise.

Apart from the division on probability, the volume is devoted mainly to geometry. However, there are about seventeen pages of text given to number theory. This work on number theory relates mainly to the solution of the equation of Pell, and closely related problems. The contributions to geometry vary in character from material as elementary as a new demonstration of the Pythagorean proposition to the determination of surfaces of parabolic and hyperbolic conoids, the tangents to algebraic curves, and the theory of evolutes. Considerable space is given to properties of the cycloid. In this work, Huygens was probably accepting the challenge put up by the letter of Pascal (under an assumed name) addressed "to all geometers of the universe" in which he proposed problems about a certain semisegment of the cycloid calling for the area, the center of gravity, volume of solids generated by revolution about certain lines, and the centers of gravity of these solids. Among the geometric contributions, we should perhaps mention especially that this volume gives properties of the logarithmic curve, and the quadrature of the hyperbola by the use of logarithms.

The division on probability is given both in Dutch and in a French translation. The text of the other divisions is, with a few exceptions, in Latin. The valuable footnotes and introductions to divisions are in French. The volume contains a table of contents, a name index, and a subject index that are very useful. When we consider the preparation of the introductions to the various divisions of the work, the large number of references, and the many footnote explanations and comments found throughout the work, it seems to the reviewer that this volume represents practically a model piece of work from the editorial standpoint as well as a contribution to the history of Huygens's mathematical work.

H. L. Rietz.

Dynamics of the Airplane. By K. P. Williams. New York, John Wiley and Sons. 136 pages.

This little volume is number 21 of the series of Mathematical Monographs edited by Mansfield Merriman and R. S. Woodward. The author states that it is an outgrowth of a set of lectures delivered at the University of Paris by Professor Marchis for the benefit of the students of the American Army.

The subject of aeronautics as usually given includes both rigid mechanics and fluid mechanics. The latter, however, is not of so much importance for elementary considerations and has been omitted from the book under consideration. The omission does not interfere with the understanding of the dynamics of the airplane even when we include Bryan's mathematical theory of stability and Wilson's treatment of the effect of gusts. Almost all mathematical problems of the airplane can be treated with no more fluid mechanics than is usually given in a good college course in physics.

The more elementary subjects usually considered in a book of this character are: pressure on a plane and curved surface, movement of the center of pressure as the inclination of the plane is varied, the relation of the velocity of flight to the angle of incidence of the wing, power consumed, ascent, descent and circular flight. These subjects are adequately and delightfully treated in the first four chapters, 68 pages. The fifth chapter is devoted to the propeller. The principle topics discussed are form of the blade, thrust, power and efficiency. The formulas obtained for these are

$$T = \alpha V^2 D^2, \qquad P = \beta n^2 D^5, \qquad E = TV/(550\, P),$$

where n denotes r.p.s., D denotes the diameter of the propeller, α and β are functions of V/nD. The last part of the chapter is devoted to the experimental determination of the functions α and β and the adaptation of the propeller to the machine. Chapter VI, on performance, discusses the highest attainable altitude and the radius of action together with the effect of loading on these. Chapter VII on stability and controlability discusses in a general way the meaning of stability and the effect of stabilizer, elevator, etc. Chapter VIII treats Bryan's theory of stability. The rigid mechanics necessary is not developed here but reference is made to Routh.

The book is well written, very attractive in appearance, and makes a splendid introduction to the subject.

C. L. E. MOORE.

The Sumario Compendioso of Brother Juan Diez: The Earliest Mathematical Work of the New World. By David Eugene Smith. Boston, Ginn and Company, 1921. vii + 65 pp.

To the historian and to the bibliophile the publication of this little volume will be an event of importance. But even to one who, like the reviewer, can lay no claim to such distinction the possession and perusal of the work will be a source of genuine interest and pleasure.

The original was printed in the City of Mexico in 1556 "long before the migration of the pilgrims to this continent was thought of, a half century before Henry Hudson discovered the river that bears his name, and nearly two centuries before the first mathematical work in English appeared from the pen of an American scholar." The author was one Juan Diez, a priest and companion of Cortez in the conquest of Mexico.

The greater part of the original consists of an elaborate set of tables intended to minimize as far as possible the computations arising in the sale of gold and silver. Only one page of these tables is reproduced in the present edition. All of the remainder of the work, however, consisting of a brief textbook on arithmetic "suited to the needs of apprentices in the counting houses of the New World" and of six pages of algebra (devoted largely to the quadratic equation), is reproduced in facsimile with an English translation appearing on the opposite pages. When it is recalled that only a small number of books on algebra had appeared in Europe before 1556 it is indeed "remarkable"—as the editor points out—"that an obscure writer in Mexico should have produced even six pages on the subject in this early period in the development of printed scientific literature."

The fact that only four copies of the original are known to exist made it desirable to make a facsimile reproduction of the work more generally available. This has been made possible through the cooperation of the distinguished editor and translator, an anonymous "public spirited gentleman" who subscribed to the greater part of the limited edition in advance of publication, and of a publishing house which has in the past given evidence not merely of high proficiency in the art of book-making but also of scholarly ideals not unduly influenced by the balance-sheet. To them the cordial thanks of the scientific fraternity are due.

J. W. Young.

Theoretische Arithmetik. Von Otto Stolz und J. A. Gmeiner. Zweite Auflage, bearbeitet von J. Anton Gmeiner. I. Abteilung, 1911, vi + 146 pp.; II. Abteilung, 1915, viii + 369 pp. Leipzig, Teubner.

Maintaining the same division into "Abschnitte" as in the first edition and largely the same separation of each into sections, Gmeiner has given us in the second edition of the *Theoretische Arithmetik* of Stolz and Gmeiner a work larger by about twenty-five percent than the first edition of one volume. In every Abschnitt one finds numerous modifications and extensions, often of a minor character but occasionally of considerable extent. Many of the modifications are in the direction indicated by Stolz in notes he had made looking forward to a revision of the text. Gmeiner has carefully utilized these suggestions so as to carry out as far as possible the wishes of his teacher in this new edition of their common work.

The larger changes may be indicated briefly as follows: There is a fuller treatment in the second Abschnitt of subtraction of integers, division, powers, and the systematic representation of numbers; and there is a new section here on zero as a number. In the third Abschnitt the theory of the laws of operation has been expounded anew and in fuller form, especially that having to do with the distributive law. The sections on the fundamental operations with real numbers in the seventh Abschnitt and the theory of complex units in the tenth Abschnitt appear in a new form. Other changes, less important, are to be found throughout the volumes. In its new form the book will have an increased usefulness.

As the work is now divided the first volume treats the theory of rational numbers and the second the theory of real and complex numbers together with an introduction to the theory of infinite series both with real terms and with complex terms.

R. D. Carmichael.

Advanced Lecture Notes on Light. By J. R. Eccles. Cambridge, University Press, 1919. 141 pp.

A sequel to the author's *Lecture Notes on Light*, 1917, the *Advanced Lecture Notes* treat in sequence Rainbows, Magnifying Power, Chromatic Aberration, Spherical Aberration, Wave Theory of Light, Interference, Diffraction, Polarisation of Light. There are really only half the indicated number of pages, because the left-hand page is left blank apparently

with the intention that the student shall draw thereon the figures which are entirely lacking in the text and perhaps make certain calculations supplementary to or illustrative of the text. A perusal of the work fails to indicate surely whether the lectures as given are largely theoretical or are amply illustrated by experiment showing the phenomena on a scale impressive even to undergraduates. No part of physics lends itself so well to treatment by experimental lectures as optics, particularly physical as contrasted with geometrical optics.

It is unnecessary to commend either the excellent care given to the treatment or the selection and order of the topics. The discussion of geometrical optics can hardly be made so moving as that of physical optics, but every pedagogic instinct, except that of fascinating or even mystifying the student, is better satisfied by commencing with the former, as the author does, and so far as the real training of the student in analysis or in the things of most use to him is concerned, geometrical optics will long remain preferable to physical. It would have been desirable, provided the students' previous training sufficed, to lay some small emphasis on the dynamics of wave motion.

The question of what shall be taught in optics and how it shall be taught is like the corresponding questions relative to mechanics, electricity, and heat, not only unsolved but as yet unstated in a form capable of solution. Eccles' book merits careful consideration by collegiate teachers of physics.

E. B. WILSON.

Materialien für eine wissenschaftliche Biographie von Gauss. Gesammelt von F. Klein, M. Brendel, und L. Schlesinger Heft VIII. *Zahlbegriff und Algebra bei Gauss.* Von A. Fraenkel. Mit einem Anhang von A. Ostrowski: *Zum ersten und vierten Gaussschen Beweise des Fundamentalsatzes der Algebra.* Erster Teil. Leipzig, Teubner, 1920. 58 pp.

The title of this pamphlet is sufficiently explanatory of its general character. As it forms only the first part of the eighth volume of the series, it is incomplete in some respects. However, it does contain an interesting and well rounded discussion of Gauss's part in the development of the concept of number and an arithmetization of his first proof of the fundamental theorem of algebra. A corresponding treatment of the fourth proof, together with detailed criticisms of the two proofs, are apparently to be given in the second part of the volume.

As the author remarks, there can be no doubt that Gauss recognized the sphere of philosophy in investigating the foundations of mathematics, and it is interesting to note that he opposed Kant's view that space is merely a creation of our senses. While Gauss's part in the development of the complex number was probably of relatively more importance, it is pointed out that he made substantial contributions to the modern theory of real numbers. The somewhat controversial subject of the priority of different writers with regard to the former seems on the whole to be treated impartially, although Argand is given less credit than some would doubtless accord him.

It is worth noting that Gauss himself recognized the somewhat unsatisfactory character of his first proof of the existence of roots of an algebraic equation. Any comments on Ostrowski's discussion would seem more appropriate in connection with Fraenkel's criticism of the original proof.

HOWARD H. MITCHELL.

ERRATA.

Vol. XXVI, p. 292, formula (22): Instead of $D=2(n-1)\cdots$ read $D = -2(n-1)\cdots$.

——, p. 293, formula (25): The denominator of the last integral in the value for x should be s^2; and, in the value for y, s^3.

Vol. XXVII, p. 11, line 6 of § 1: Instead of the words *of all functions* read *of all bounded functions*.

——, p. 11, formula (2): Add after the formula $(a < y < b)$.

——, p. 17, line 17: Add after the last sentence the sentence: In order to make sure that the integral (1) shall belong to the class $[f]$, it is necessary to assume also that $K(x, y) - K(x, a)$, considered as a function of x, belongs to that class for every value of y.

——, p. 326: Professor R. L. Borger desires to withdraw, at least tentatively, the theorem he announced on this page, on account of a flaw in the proof near the bottom of page 327, which was called to his attention by Dr. T. H. Gronwall.

——, p. 364, line 17: Instead of $m = -9$ read $m = -91$.

——, p. 385, last line: Instead of Granier read Garnier.

NOTES.

At the request of the Council of the Society, the Committee of Publication has decided to begin the next volume of this Bulletin (vol. XXVIII) with the number to be dated January, 1921. Hence the usual October, November, December numbers will not be issued this fall. Volume XXVIII will consist of ten numbers as usual, however, with mere change of dates. This change of dates has been contemplated for several years; — it will be convenient because the Bulletin year will hereafter coincide with the calendar year, with the fiscal year of the Society, and with the terms of office of the editors and other officers. This year the change can be made without any real delay, for the printers' strike has so far deferred publication that to date the next number in January will only correspond to the actual fact. Subscribers to the new volume should notice that they will receive the usual number of issues, and that they will actually receive them quite as soon as would be possible otherwise.

The annual meeting of the Society will be held in Toronto on December 28–29, in connection with the sessions of the American Association for the Advancement of Science. The usual Christmas meeting at Chicago, as well as the annual meeting usually held in New York, has been transferred to Toronto, and will be merged with the annual meeting. Titles and abstracts of papers for this meeting should be sent to the Secretary of the Society, R. G. D. Richardson, Brown University, and must be in his hands not later than December 10, 1921.

At one of the sessions of the Toronto meeting, Professor R. D. Carmichael will give an address as retiring chairman of the Chicago Section. The subject will be *Algebraic guides to transcendental problems*.

The fourteenth regular meeting of the Southwestern Section of the Society will be held in St. Louis at Washington University on Saturday, November 25. Titles and abstracts of papers for this meeting must be in the hands of the Secretary of the Section, E. B. Stouffer, University of Kansas, not later than November 5, 1921.

A list of dates of meetings and other announcements is printed on the inside of the back cover.

The April number (vol. 22, no. 2) of the TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY contains the following papers: *On division algebras*, by J. H. M. Wedderburn; *Oscillation theorems for the real, self-adjoint linear system of the second order*, by H. J. Ettlinger; *New proofs of certain finiteness theorems in the theory of modular covariants*, by Olive C. Hazlett; *On the convergence of certain trigonometric and polynomial approximations*, by Dunham Jackson; *Determination of all general homogeneous polynomials expressible as determinants with linear elements*, by L. E. Dickson; *Pseudo-canonical forms and invariants of systems of partial differential equations*, by A. L. Nelson; *Arithmetical paraphrases* (II), by E. T. Bell; *On the zeros of solutions of homogeneous linear differential equations*, by C. N. Reynolds, Jr.; *A generalization of the Fourier cosine series*, by J. L. Walsh; *Polynomials and their residue systems*, by A. J. Kempner.

The April number (vol. 43, no. 2) of the AMERICAN JOURNAL OF MATHEMATICS contains: *Boundary value and expansion problems: algebraic basis of the theory*, by R. D. Carmichael; *Algebraic theory of the expressibility of cubic forms as determinants, with application to diophantine analysis*, by L. E. Dickson; *The impossibility of Einstein fields immersed in flat space of five dimensions*, by Edward Kasner; *Finite representation of the solar gravitational field in flat space of six dimensions*, by Edward Kasner; *On the motion of two spheroids in an infinite liquid along their common axis of revolution*, by Bibhutibhusan Datta.

The mathematical seminar of the University of Warsaw has founded a new periodical, FUNDAMENTA MATHEMATICÆ, devoted entirely to the theory of assemblages and related subjects, such as analysis situs, mathematical logic and theory of axioms. Articles offered for publication should be sent to the editors, S. Mazurkiewicz and W. Sierpiński, at the University of Warsaw, and must be written in French, Italian, English, or German. Two volumes have already appeared.

The Society of German Engineers announces that it will publish, beginning early in 1921, a new periodical devoted to applied mathematics, the ZEITSCHRIFT FÜR ANGEWANDTE MATHEMATIK UND MECHANIK. The editors are R. von Mises (editor in chief), A. Föppl, R. Mollier, H. Müller-Breslau, L. Prandtl, and R. Rüdenberg.

Attention is called to the announcement, on one of the fly-leaves of this number of the BULLETIN, concerning the JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES, often called LIOUVILLE'S JOURNAL or JORDAN'S JOURNAL. The success of the effort to maintain this important journal is of interest to mathematicians everywhere. Gratifying response has been received already by friends of the JOURNAL in this country.

The summer meeting of the British Association for the Advancement of Science will be held at Edinburgh, September 7–14. Professor O. W. Richardson has been appointed president of Section A (mathematics and physics) for this meeting.

A meeting of the National Academy of Sciences was held in Washington, April 25, 26 and 27, 1921. Professor Albert Einstein and the Prince of Monaco were guests of honor at the annual dinner. Professor G. A. Miller and Professor A. E. Kennelly were elected to membership. The following mathematical papers were read: by Edward Kasner, *A model of the solar gravitational field;* by G. D. Birkhoff, *On the problem of three or more bodies;* by L. E. Dickson, *Quaternions and their generalizations;* by L. E. Dickson, *Investigations in algebra and number theory;* by H. F. Blichfeldt, *On the approximate solution in integers of a set of linear equations.*

At a meeting of the American Philosophical Society in Philadelphia, April 21, 22 and 23, Professor G. D. Birkhoff was elected to membership. The following papers were read by members of the American Mathematical Society: by L. E. Dickson, *The atomic theory and ideal numbers;* by J. A. Miller, *Application of the interferometer to the determination of stellar parallaxes;* by F. R. Moulton, *Recent astronomical explorations in space and in time;* by M. B. Snyder, *1. Universal volcanism and the cosmic atomic numbers; 2. Planck's constant "h," a variable.*

As a result of the preliminary conference held in January (BULLETIN, p. 336) a committee of the National Research Council on the mathematical analysis of statistics has been organized, consisting of H. L. Rietz (chairman), J. W. Glover, E. V. Huntington, T. L. Kelley, R. Pearl, and W. M. Persons.

The lectures on the theory of relativity delivered by Professor Albert Einstein at Princeton (BULLETIN, p. 386) will be published by the Princeton University Press. Professor Einstein lectured at the University of Chicago on May 3, 4 and 5.

Professor Emile Borel has been elected a member of the Paris Academy of Sciences in the section of geometry, as successor to the late Professor Georges Humbert. Sir George Greenhill has been elected correspondent in the section of mechanics, as successor to the late Professor W. Voigt.

Professor Albert Einstein has been elected a foreign member of the Royal Society of London.

The honorary degree of doctor of science has been conferred by the University of Manchester on Dr. Horace Lamb and Sir Ernest Rutherford.

The Royal Society of Edinburgh has awarded the Macdougall-Brisbane prize for the period 1918–1920 to Professor J. H. M. Wedderburn of Princeton, in recognition of his investigations on hypercomplex numbers.

Professor E. Naetsch has been appointed professor of analytic geometry at the Technical School at Dresden, not at the Technical School at Darmstadt, as was announced in the May number of this BULLETIN.

Dr. G. Bouligand, of the University of Rennes, has been appointed professor of rational mechanics at the University of Poitiers.

Sir Asutosh Mookerjee (Mukhopâdhyây), president of the Calcutta Mathematical Society, has been appointed vice-chancellor of Calcutta University.

At a general meeting of the members of the Royal Institution held May 9, Sir J. J. Thomson was elected honorary professor of natural philosophy, and Sir Ernest Rutherford professor of natural philosophy.

Mr. T. A. Brown has been appointed senior lecturer in mathematics for 1921–1922 at University College, London.

It is reported that Professor H. Lamb is to be appointed to an honorary lectureship to be called the Rayleigh lectureship in mathematics, at Cambridge University.

Arrangements for exchange professorships in engineering and applied sciences have recently been completed between certain French and American universities and technical schools. Professor A. E. Kennelly, of Harvard University and the Massachusetts Institute of Technology, has been chosen as the first American representative.

One of the recently endowed Sterling professorships at Yale University has been assigned to Professor E. W. Brown.

Assistant Professor P. J. Daniell, of the Rice Institute, has been promoted to a full professorship of applied mathematics.

Assistant Professor H. M. Robert, Jr., of the United States Naval Academy, has been promoted to an associate professorship of mathematics.

At the University of Alberta, Associate Professor S. D. Killam has been promoted to a full professorship and Assistant Professor J. W. Campbell to an associate professorship.

At Cornell University, Professor James McMahon retires at the close of the current academic year. The previously announced Heckscher Research grant to Mr. H. S. Vandiver for the continuation of his investigations on the theory of algebraic numbers has been extended to cover the first term of the year 1921–22, and Mr. Vandiver has been granted leave of absence for that period. Mr. Jesse Osborn of Pennsylvania State College and Mr. R. L. Jeffery of Acadia College have been appointed instructors.

At Clark University, Professors W. E. Story, Henry Taber and Joseph de Perott retired in June.

At Dartmouth College, Professor C. N. Haskins has been appointed chairman of the department of mathematics for a term of two years. Assistant Professor R. D. Beetle has been promoted to a full professorship. Professor E. G. Bill has been appointed Dean of Freshmen; he retains his professorship, but will not give instruction during the academic year 1921–22. Professor J. W. Young returns to the department after two years' leave of absence during which he has served as chairman of the National Committee on Mathematical Requirements. Assistant Professor F. M. Morgan has been granted leave of absence for the second semester of the academic year 1921–22.

Dr. J. F. Ritt, of Columbia University, has been promoted to an assistant professorship of mathematics.

Mr. H. L. Smith, of the University of Wisconsin, has been appointed professor of mathematics at the University of the Philippines.

Dean P. C. Porter, of the Baylor College has been appointed assistant professor of mathematics at Baylor University.

Associate Professor E. D. Grant, of the Michigan College of Mines, has been appointed head of the department of mathematics at Earlham College.

Mr. J. A. Herrington, of the University of Wisconsin, has been appointed associate professor of mathematics and engineering at Beloit College.

Mr. T. L. Hamlin, of Union College, has been appointed head of the department of mathematics at the Clarkson College of Technology, Potsdam, N. Y., as successor to Professor H. M. Royal, resigned.

At the College of Wooster, Mr. C. O. Williamson has been promoted to an assistant professorship of applied mathematics.

Assistant Professor F. N. Bryant, of the State College of Washington, has been appointed assistant professor of mathematics at Syracuse University.

At the College of William and Mary, Mr. J. S. Counselman has been appointed professor of mathematics and Mr. J. C. Lyons assistant professor.

Professor J. B. Hamilton has been appointed head of the department of mathematics at the University of Tennessee.

Mr. J. P. Beaman has been appointed head of the department of mathematics at the Kentucky Wesleyan College, as successor to Professor C. S. Venable, resigned.

Mr. F. J. Rogers has been appointed head of the department of mathematics at Shaw University, Raleigh, N. C.

Professor L. S. Dancey has been appointed head of the department of mathematics at Carroll College, Waukesha, Wis., as successor to Professor S. B. Ray, resigned.

Mr. L. C. Bagley, of Washburn College, has been appointed professor of mathematics at Ottawa University.

Mr. Thurman Andrew has been promoted to a professorship at the West Virginia Wesleyan College, as successor to Professor C. E. White, resigned.

In October, 1920, Mr. F. A. Lewis, of the Texas Agricultural and Mechanical College, and Dr. C. G. P. Kuschke were appointed assistant professors of mathematics.

At The Citadel, the military college of South Carolina, Major R. G. Thomas, professor of mathematics and engineering for thirty-one years, has been made professor emeritus. He was succeeded by Colonel O. J. Bond.

At Fairmount College, Wichita, Mr. H. G. Titt, of Yankton College, has been appointed professor of mathematics and dean of the College of Liberal Arts.

At Villanova College, Reverend J. S. O'Leary has been appointed professor of mathematics, as successor to Reverend F. A. Driscoll, who has been made president of the college.

Mr. E. K. Paxton has been appointed assistant professor of mathematics at Washington and Lee University.

Professor A. H. Norton, of Elmira College, has been appointed president and head of the department of mathematics at the Keuka College for Women.

Professor E. F. Canaday, of the University of South Dakota, has been appointed professor of mathematics at Meredith College.

At Central Wesleyan College, Warrenton, Mo., Professor J. H. Frick, head of the department of mathematics for fifty years, has retired. He was succeeded by Mr. H. V. Knorr.

Dr. V. H. Wells has been appointed assistant professor of mathematics at Carleton College.

At the University of Oklahoma, Dr. J. O. Hassler, of the Crane Junior College, has been appointed associate professor of mathematics. Assistant Professor H. C. Gossard has resigned, to enter Y. M. C. A. work Assistant Professor E. D. Meacham has been granted leave of absence for 1921–22.

Mr. F. H. Murray, of Harvard University, has been awarded one of the American Field Service fellowships for study in the French universities.

Recent appointments as instructor in mathematics in American colleges and universities are as follows: Colgate University, Mr. Torvald Frederiksen; Columbia University, Dr. E. L. Post; Cornell College, Mt. Vernon, Iowa, Mr. G. F. Rouse; Denison University, Miss Grace E. Jefferson; Illinois College, Miss Lois Daniels; Lafayette College, Mr. R. J. W. Templin; Princeton University, Mr. Philip Franklin; Randolph-Macon College, Mr. B. F. Walton; Randolph-Macon Woman's College, Miss Virginia Watts; Ripon College, Mr. William Bollenbeck; Shorter College, Miss Jennie D. Ehlers; State College of Washington, Miss Gladys H. Freeman and Mr. H. H. Irwin; Tufts College, Miss Edith L. Bush and Mr. B. A. Hazeltine; University of Maine, Mr. A. I. Bless; University of Nebraska, Mr. J. H. Taylor; University

of Oklahoma, Mr. H. G. Lieber, Miss Dora McFarland and Miss Ella Mansfield; University of Pittsburgh, Mr. F. J. Burkett; University of Vermont, Mr. F. W. Householder and Mr. H. G. Millington.

From an account of the life and work of Staeckel sent by Professor W. Lorey, the following extracts are selected:* Paul Staeckel was born at Berlin, August 20, 1862, and died December 12, 1919. He studied at the University of Berlin, having among his teachers Kronecker, Weierstrass, Helmholtz and Fuchs; he received his doctorate in 1884 and his teacher's certificate in 1885. He taught at the Berlin Wilhelm gymnasium and at the universities and technical schools of Halle, Königsberg, Hanover, Karlsruhe and Heidelberg. Among his scientific researches are studies on mechanics, arithmetic properties of analytic functions and the representation of even numbers as sums and differences of primes. He edited the fourth volume of the German report of the International Commission on the Teaching of Mathematics, translated Borel's *Eléments de Mathématiques*, and exhibited elsewhere much interest in pedagogy. His contributions to the history of mathematics include editorial service in connection with Euler's collected works and with Ostwald's mathematical classics; a collection of original sources on the theory of parallels, in collaboration with Engel; a study of the geometric work of Gauss; and numerous articles in Bibliotheca Mathematica.

Professor A. M. G. Floquet, of the University of Nancy, died October 7, 1920, at the age of seventy-two years.

Colonel Vallier, correspondent of the Paris Academy of Sciences in the section of mechanics and well known for his work in ballistics, died March 29, 1921.

Professor Alfred Doolittle, of the Catholic University, died February 23, 1921.

Mr. J. S. Hoffman, instructor in mathematics at Tufts College, died June 29, 1920, at the age of twenty-three years.

* A full account by Professor Lorey appears in the ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT ALLER SCHULGATTUNGEN, vol. 52, pp. 85–88.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BERZOLARI (L.). Geometria analitica. I: Il metodo delle coordinate. 2a edizione. (Manuali Hoepli.) Milano, Hoepli, 1920. 16mo. 14 + 496 pp.

BHATTACHARYYA (D.). Vector calculus. (Griffith Prize thesis.) Calcutta, University Press, 1920. 8vo. 90 pp.

BIBLIOTHECA chemico-mathematica: catalogue of works in many tongues on exact and applied science, with a subject index. Compiled and annotated by H. Zeitlinger and H. C. S. London, Sotheran, 1921. 2 volumes. 8vo. 12 + 964 pp. £ 3 3s.

CARNOT (L.). Réflexions sur la métaphysique du calcul infinitésimal. 2 volumes. (Les Maîtres de la Pensée Scientifique.) Paris, Gauthier-Villars, 1921. 16mo. 8 + 117 + 105 pp. Fr. 3.50 + 3.50

CLAIRAUT (A. C.). Eléments de géométrie. 2 volumes. (Les Maîtres de la Pensée Scientifique.) Paris, Gauthier-Villars, 1920. 16mo. 14 + 95 + 103 pp. Fr. 3.50 + 3.50

DIEZ (J.). See SMITH (D. E.).

ENGEL (F.). See EULER (L.).

EULER (L.). Opera omnia. Series I: Opera mathematica. Leipzig, Teubner. II–III: Commentationes arithmeticæ. Edidit F. Rudio. Volumen 1, 38 + 611 pp., 1915. Volumen 2, 38 + 543 pp., 1917. XIII: Institutiones calculi integralis. Ediderunt F. Engel et L. Schlesinger. Volumen 3, 18 + 508 pp., 1914. XVII–XVIII: Commentationes analyticæ ad theoriam integralium pertinentes. Volumen 1, edidit A. Gutzmer, 8 + 467 pp., 1915. Volumen 2, ediderunt A. Gutzmer et A. Liapounoff, 12 + 475 pp., 1920.

FUBINI (G.) e VIVANTI (G.). Esercizi di analisi matematica. (Calcolo infinitesimale.) (Grande Biblioteca Tecnica, No. 13.) Torino, Società Tipografico-Editrice Nazionale, 1920. 8vo. 8 + 575 pp. L. 58.00

GATES (S. B.). Pure mathematics for engineers. With an introduction by H. A. Webb. 2 volumes. London, Hodder and Stoughton, 1920. 11 + 191 + 11 + 179 pp. 4s. 6d. + 4s. 6d.

GUTZMER (A.). See EULER (L.).

HAHN (H.). Theorie der reellen Funktionen. 1ter Band. Berlin, Springer, 1921. 8vo. 8 + 600 pp.

HALER (P. J.) and STUART (A. H.). A second course in mathematics for technical students. London, University Tutorial Press, 1920. 8 + 363 pp. 6s.

HEFFTER (L.). Die Grundlagen der Geometrie als Unterbau für die analytische Geometrie. Leipzig, Teubner, 1921. 8vo. 27 pp.

H. C. S. See BIBLIOTHECA.

KOMMERELL (V.) und KOMMERELL (K.). Allgemeine Theorie der Raumkurven und Flächen. 1ter Band. 2ter Band. 3te Auflage. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1921. 8 + 184 pp. 2 + 196 pp. $1.75

LECAT (M.). Pensées sur la science, la guerre, et sur des sujets très variés. Bruxelles, Lamertin, 1919. Royal 8vo. 7 + 478 pp. Fr. 32.00

LIAPOUNOFF (A.). See EULER (L.).

MANNING (W. A.). Primitive groups. Part 1. (Stanford University Publications, University Series, Mathematics and Astronomy, vol. 1, No. 1.) Stanford University, 1921. 8vo. 108 pp. $1.25

MIELI (A.). Gli scienziati italiani dall' inizio del medio evo ai nostri giorni. Diretto da A. Mieli. Volume 1, parte 1. Roma, Nardecchia, 1921. Sm. 4to. 8 + 236 pp. L. 45.00

MILNE (W. P.) and WESTCOTT (G. J. B.). A first course in calculus. Part 2: Trigonometric and logarithmic functions of x, etc. London, Bell, 1920. 15 + 222 + 24 pp. 5s.

MORDELL (L. J.). Three lectures on Fermat's last theorem. Cambridge, University Press, 1921. 3 + 31 pp. 4s.

OSGOOD (W. F.). Elementary calculus. New York, Macmillan, 1921. 12mo. 9 + 224 pp. $2.40

RUDIO (F.). See EULER (L.).

SCHLESINGER (L.). See EULER (L.).

SCHNEIDER (H.). Metaphysik als exakte Wissenschaft. Heft I: Gegebenheitslehre. Heft II: Gegliedertheitslehre. Leipzig, Hinrichssche Buchhandlung, 1919–1920. 4 + 143 + 2 + 189 pp.

SMITH (D. E.). The Sumario compendioso of Brother Juan Diez. The earliest mathematical work of the new world. Boston, Ginn, 1921. 8vo. 65 pp. $4.00

STUART (A. H.). See HALER (P. J.).

STUYVAERT (M.). Congruences de cubiques gauches. Louvain, Van Rysselberghe et Rombaut, 1920. 8vo. 200 pp. Fr. 12.50

VIVANTI (G.). See FUBINI (G.).

VAN WEEL (D. E. W.). De oplossing der trisectie. Leeuwarden, Meijer en Schaafsma, 1919. 23 pp. f 0.60

WEBB (H. A.). See GATES (S. B.).

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II. ELEMENTARY MATHEMATICS.

EDWARDS (R. W. K.). An elementary text-book of trigonometry. London, Harrap, 1921. 13 + 251 pp. 5s.

HAWKES (H. E.), LUBY (W. A.) and TOUTON (F. C.). Plane geometry. Boston, Ginn, 1920. 8 + 305 pp. $1.32

HENSON (J. W.). Solid geometry. London, Blackie, 1920. 7 + 88 pp. 3s.

JACKSON (D.). Specimen answers of college candidates in plane geometry written at the examinations in June, 1920. (College Entrance Examination Board, Document No. 99.) New York, College Entrance Examination Board, 1921. 22 pp. $0.25

KAYE (G. W. C.) and LABY (T. H.). Tables of physical and chemical constants and some mathematical functions. 4th edition. New York, Longmans, 1921. 8vo. 8 + 161 pp. $4.00

LABY (T. H.). See KAYE (G. W. C.).

LUBY (W. A.). See HAWKES (H. E.).

MOYER (J. A.) and SAMPSON (C. H.). Practical trade mathematics. New York, Wiley, 1920. 9 + 172 pp. $1.50

OVERMAN (J. R.). Principles and methods of teaching arithmetic. Chicago, Lyons and Carnahan, 1920. 6 + 350 pp. $1.60

PIGAL (H.). Méthode pratique de règle à calcul type Mannheim. Paris, Librairie Desforges, 1921. 8vo. 114 pp. Fr. 7.50

SAMPSON (C. H.). See MOYER (J. A.).

TOUTON (F. C.). See HAWKES (H. E.).

III. APPLIED MATHEMATICS.

ADAMS (M.). See GARNETT (W.)

ANNUAIRE pour l'an 1921 publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1921. 16mo. 7 + 710 + 130 pp. Fr. 6.00

DE BELLESCIZE (H.). Etude de quelques problèmes de radiotélégraphie. Paris, Gauthier-Villars, 1920. 8vo. 8 + 174 pp. Fr. 16.00

BIGOURDAN (G.). Bibliothèque bibliographique et documentaire. Section des sciences pures et appliquées. 3e partie: Astronomie, géodésie et géophysique. Paris, Gauthier-Villars, 1920. 8vo. 257 pp. Fr. 20.00

BOGGIO (T.). See BURALI-FORTI (C.).

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BOULVIN (J.). Calcul des organes des machines. Paris, Gauthier-Villars-1920. 8vo. 10 + 516 pp.

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CALLENDAR (H. L.). Properties of steam and thermodynamic theory of turbines. New York, Longmans, 1921. 8vo. 12 + 531 pp. $14.00

CLERC (L. P.). Applications de la photographie aérienne. (Encyclopédie Scientifique.) Paris, Doin, 1920. 6 + 350 + 12 pp. Fr. 7.50

COLE (R. V.). The practice and theory of perspective as applied to pictures, with a section dealing with its application to architecture. London, Seeley, Service and Company, 1921. 18s.

CUNNINGHAM (E.). Relativity, the electron theory, and gravitation. 2d edition. New York, Longmans, 1921. 8vo. 8 + 148 pp. $3.50

DALE (G. H. C.). Map reading. London, Macmillan, 1921. 9 + 170 pp. 7s. 6d.

DITTLER (R.). Stereoskopisches Sehen und Messen. Leipzig, Barth, 1919. 36 pp. M. 3.00

EINSTEIN (A.). L'éther et la théorie de la relativité. Traduit de M. Solivine. Paris, Gauthier-Villars, 1921. 16 pp. Fr. 2.50

——. La théorie de la relativité restreinte et généralisée. Traduction de Mlle. J. Rouvière. Préface de E. Borel. Paris, Gauthier-Villars, 1921. 16mo. 22 + 120 pp. Fr. 7.00

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EYDOUX (D.). Hydraulique générale et appliquée. (Encyclopédie du Génie civil et des Travaux publics.) Paris, J. B. Baillière, 1921. 8vo. 512 pp. Fr. 40.00

FABRE (L.). Les théories d'Einstein. (Une nouvelle figure du monde). Paris, Payot, 1921. 16mo. 242 pp. Fr. 7.50

GARNETT (W.) [Mary Adams, pseud.] A little book on map projection. New and revised edition. London, G. Philip and Son, 1920. 8 + 112 pp. 5s. 6d.

KANTHACK (R.). See VON ROHR (M.).

LANDON (J. W.). Elementary dynamics: a text-book for engineers. Cambridge, University Press, 1920. 8 + 246 pp. 10s. 6d.

DE LAPLACE (P. S.). See LAVOISIER (A. L.).

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LAVOISIER (A. L.) et DE LAPLACE (P. S.). Mémorie sur la chaleur. (Les Maîtres de la Pensée Scientifique.) Paris, Gauthier-Villars, 1920. 16mo. 78 pp. Fr. 3.00

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MAXWELL (J. C.). Matter and motion. Reprinted with notes and appendices by Sir Joseph Larmor. London, Society for Promoting Christian Knowledge, and New York, Macmillan, 1920. 15 + 163 pp. 5s.

VAN MOLLÉ (J.). Les théories physiques. 2e édition. Malines, H. Dierickx-Becke fils, 1921. 8vo. 8 + 360 pp. Fr. 8.00

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VON ROHR (M.). Geometrical investigation of the formation of images in optical instruments. (Volume 1 of The theory of optical instruments.) Translated by R. Kanthack. London, published for the Department of Scientific and Industrial Research by His Majesty's Stationery Office, 1920. 23 + 612 pp. 2£ 5s.

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DE VILLAMIL (R.). Soaring flight. A simple mechanical solution of the problem. London, Spon, 1920. 48 pp. 1s. 6d.

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ALEXANDER, J. W. On the equilibrium of a fluid mass at rest. Read Feb. 28, 1920. *Transactions of the American Mathematical Society*, vol. 21, No. 4, pp. 446–450; Oct., 1920.

ALTSHILLER-COURT, N. Sur la cubique à point double. Read April 26, 1913. *Nouvelles Annales de Mathématiques*, ser. 4, vol. 20, No. 11, pp. 424–435; Nov., 1920.

—— On a pencil of nodal cubics. Second paper. Read April 24, 1920. *Bulletin of the American Mathematical Society*, vol. 27, No. 3, pp. 114–115; Dec., 1920.

BARNETT, I. A. Functionals invariant under one-parameter continuous groups of transformations in the space of continuous functions. Read April 24, 1920. *Proceedings of the National Academy of Sciences*, vol. 6, No. 4, pp. 200–204; April, 1920.

BELL, E. T. Sur les représentations propres par quelques formes quadratiques de Liouville. Read (San Francisco) Oct. 25, 1919. *Journal de Mathématiques Pures et Appliquées*, ser. 8, vol. 2, pp. 249–271; 1919.

—— Parametric solutions for a fundamental equation in the general theory of relativity, with a note on similar equations in dynamics. Read (San Francisco) April 10, 1920. *Philosophical Magazine*, ser. 6, vol. 39, No. 231, pp. 285–288; March, 1920.

—— On a certain inversion in the theory of numbers. Read (San Francisco) Oct. 25, 1919. *Tôhoku Mathematical Journal*, vol. 17, Nos. 3–4, pp. 221–231; May, 1920.

—— On the representations of numbers as sums of 3, 5, 7, 9, 11 and 13 squares. Read (San Francisco) April 5, 1919. *American Journal of Mathematics*, vol. 42, No. 3, pp. 168–188; July, 1920.

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—— An image in four-dimensional lattice space of the theory of the elliptic theta functions. Read (San Francisco) June 18, 1920. *Bulletin of the American Mathematical Society*, vol. 27, No. 4, pp. 153–160; Jan., 1921.

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—— Arithmetical paraphrases (II). Read (San Francisco) Dec. 14, 1918. *Transactions of the American Mathematical Society*, vol. 22, No. 2, pp. 198–219; April, 1921.

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INDEX OF VOLUME XXVII.

Reviews.

Notes and Other Items.

Personal Notes: